A Generalized Processor Sharing Approach to Flow Control in Integrated Services Networks: The Multiple Node Case *

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Abstract

Worst-case bounds on delay and backlog are derived for leaky bucket constrained sessions in arbitrary topology networks of Generalized Processor Sharing (GPS) [10] servers. The inherent flexibility of the service discipline is exploited to analyze broad classes of networks. When only a subset of the sessions are leaky bucket constrained, we give succinct per-session bounds that are independent of the behavior of the other sessions and also of the network topology. However, these bounds are only shown to hold for each session that is guaranteed a *backlog clearing rate* that exceeds the token arrival rate of its leaky bucket.

A much broader class of networks, called Consistent Relative Session Treatment (CRST) networks is analyzed for the case in which **all** of the sessions are leaky bucket constrained. First, an algorithm is presented that characterizes the internal traffic in terms of average rate and burstiness, and it is shown that all CRST networks are stable. Next, a method is presented that yields bounds on session delay and backlog given this internal traffic characterization. The links of a route are treated collectively, yielding tighter bounds than those that result from adding the worst-case delays (backlogs) at each of the links in the route. The bounds on delay and backlog for each session are efficiently computed from a *universal service curve*, and it is shown that these bounds are achieved by "staggered" greedy regimes when an *independent sessions relaxation* holds. Propagation delay is also incorporated into the model.

Finally, the analysis of arbitrary topology GPS networks is related to Packet GPS networks (PGPS). The PGPS scheme was first proposed by Demers, Shenker and Keshav [5] under the name of Weighted Fair Queueing. For small packet sizes, the behavior of the two schemes is seen to be virtually identical, and the effectiveness of PGPS in guaranteeing worst-case session delay is demonstrated under certain assignments.

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1 Introduction

The problem of providing performance guarantees to the diverse users of an integrated services network is central to supporting real-time services such as voice and video. This problem is especially difficult in the presence of congestion, when it is important to use the link bandwidth efficiently. In [10] we proposed the combination of leaky bucket admission control and a work-conserving packet service discipline at the nodes of the network, to accommodate the delay and throughput requirements of a wide range of co-existing sessions. The service discipline is based on Generalized Processor Sharing (GPS) and was first proposed in [5] under the name of Weighted Fair Queueing (and which is discussed extensively in [10]). The emphasis in [5] was on treating the users equally; our focus is on the inherent flexibility of the mechanism, and on providing good network-wide per-session bounds on worst-case delay and backlog. In [10] we analyzed single node systems; here we extend this analysis to arbitrary topology networks of GPS servers, and relate the GPS results to networks in which the nodes follow the Packet GPS (PGPS) service discipline.

A GPS server, m, that serves N sessions on a link is characterized by N positive real numbers, $\phi_1^m, \phi_2^m, ..., \phi_N^m$. These numbers denote the relative amount of service to each session in the sense that if $S_i^m(\tau, t)$ is defined as the amount of session i traffic served by server m during an interval $[\tau, t]$, then

$$\frac{S_i^m(\tau, t)}{S_j^m(\tau, t)} \geq \frac{\phi_i^m}{\phi_j^m}, \quad j = 1, 2, ..., N \tag{1}$$

for any session i that is continuously backlogged in the interval $[\tau, t]$. A session is backlogged at time t if a positive amount of that session's traffic is queued at time t. Thus (1) is satisfied with equality for two sessions i and j that are both backlogged during the interval $[\tau, t]$.

Note from (1) that whenever session i is backlogged it is guaranteed a minimum service rate of

$$g_i^m = \frac{\phi_i^m}{\sum_{j=1}^N \phi_j^m} r^m, \tag{2}$$

where r^m is the rate of the link represented by node m. This rate is called the *session i backlog clearing rate* since a session i backlog of size q is served in at most $\frac{q}{g_i^m}$ time units.

We assume a virtual circuit, connection-based packet network, and analyze the performance of leaky bucket constrained sessions. The session i leaky bucket is characterized by a token bucket of size σ_i and a token arrival rate of ρ_i. The amount of session i traffic entering the network during any interval $(\tau, t]$ is defined to be $A_i(\tau, t)$; if session i is leaky bucket constrained, then

$$A_i(\tau, t) \leq \sigma_i + \rho_i(t - \tau), \quad \forall t \geq \tau \geq 0. \tag{3}$$

As in [10], we say that A_i conforms to (σ_i, ρ_i), or $A_i \sim (\sigma_i, \rho_i)$. For details on how to accommodate peak rate constraints as well, see [9]. The constraint (3) is identical to the one suggested by Cruz [3].

The main question we address in this paper is the following: Given a network with the values of the server parameters fixed and a set of leaky bucket constrained sessions, what is the worst-case session delay and backlog for each of the sessions in this set?

Our approach is tailored to provide bounds for the GPS and PGPS service disciplines by exploiting properties derived in [10]. We develop a methodology that applies to *arbitrary* topology networks. For a given session, our bounds consider the route of a session *as whole*, which allows for much tighter bounds than acheivable by adding worst-case delays at each hop of the route. It is important to obtain such bounds for two reasons. First, the bounds can form the basis of delay guarantees to real-time traffic, and second, the generality of the GPS service discipline allows one to examine the behavior of the network (in the worst case) under a wide range of strategies for supporting multimedia traffic. These strategies translate into the assignments of the the GPS servers (the ϕ_i's) and the parameters of the leaky bucket.

Similar issues have been addressed by the recent and important work in [4, 7, 1, 12]. The pioneering work of Cruz in evaluating multihop bounds [4] has been most useful to us. However, his results do not hold for non-acyclic network topologies and do not consider the service disciplines of interest, GPS and PGPS. Also, the bounds derived do not incorporate the inter-hop dependencies involved along a given session's route and are computed by adding worst-case delays at each hop. In [7, 1, 12], bounds are computed for multihop networks under *distributional* constraints on the input traffic. There is much merit to this approach, but most of the bounds obtained prior to our work have been shown only for acyclic networks, and further the methodology adopted makes it difficult to distinguish among the performance of various service disciplines. Thus for specific service disciplines, the bounds obtained can be quite weak. The bounds are obtained by essentially *adding* the worst-case bounds for each hop considered in isolation (in [1] Holder's inequality is used), an approach which may result in loose bounds when service disciplines such as GPS and PGPS are employed. Promising recent work by Yaron and Sidi, [13], has extended the results of this paper to obtain bounds for the EBB distributional model of [12].

In Section 2 we set up our model of the network and specify notation. Then the notions of network backlog and delay are discussed and graphically interpreted. Section 4 contains succinct per-session bounds for the leaky bucket constrained sessions of a network, which are independent of the topology and of the behavior of other sessions. Next, we treat the case when all of the sessions are leaky bucket constrained. An important tool for the analysis, the All-Greedy bound, is presented in Section 6. In Section 7, an algorithm is derived that enables a characterization of internal traffic in terms of burstiness, average and peak rates for a broad class of server allocations called Consistent Relative Session Treatment (CRST) assignments. This class of assignments is flexible enough to accommodate a wide variety of session delay constraints. In Section 8, we show that worst-case session delay and backlog can be bounded from an easily computable universal service curve. This is accomplished even though *different* worst-case regimes may maximize delay and backlog for a given session. The bounds are shown to be tight under an independent relaxation assumption, when the traffic follows a *staggered greedy* regime. Propagation delay is included in Section 9. In Section 10 our results are related to the case of PGPS networks. This extension is important since GPS is not a realizable service discipline, and its relationship (in terms of performance) to PGPS in arbitrary topology networks must be established. Conclusions are in Section 11.

2 The Network Model

The network is modeled as a directed graph in which nodes represent switches and arcs represent links. A route is a path in the graph, and the path taken by session i is defined as $P(i)$. Let $P(i,k)$ be the k^{th} node in $P(i)$, and K_i be the total number of nodes in $P(i)$. The rate of the link associated with server m is denoted by r^m.

The amount of session i traffic that enters the network in the interval $[\tau, t]$ is given by $A_i(\tau, t)$. Let $S_i^{(k)}(\tau, t)$, $k = 1, \ldots, K_i$, be the amount of session i traffic served by node $P(i,k)$ in the interval $[\tau, t]$. Thus, $S_i^{(K_i)}$ is the traffic that leaves the network. We characterize the service function by "pseudo" leaky bucket parameters $\sigma_i^{(k)}$ and ρ_i so that

$$S_i^{(k)}(\tau, t) \leq \sigma_i^{(k)} + \rho_i(t - \tau), \quad \forall t \geq \tau \geq 0, \tag{4}$$

i.e., $S_i^{(k)} \sim (\sigma_i^{(k)}, \rho_i)$.

Often, we will analyze what happens at a particular server, m. In this case the notation described above becomes overly cumbersome. Define $I(m)$ to be the set of sessions that are served by server m. For every session $i \in I(m)$, let the arrival function into that node be described by $A_i^m \sim (\sigma_i^m, \rho_i)$ and the departure function be described by $S_i^m \sim (\sigma_i^{m,out}, \rho_i)$. For example, at server 0 in Figure 1: $A_0^0 = A_0$, $A_2^0 = S_2^{(2)}$, and $A_3^0 = S_3^{(1)}$. Thus when

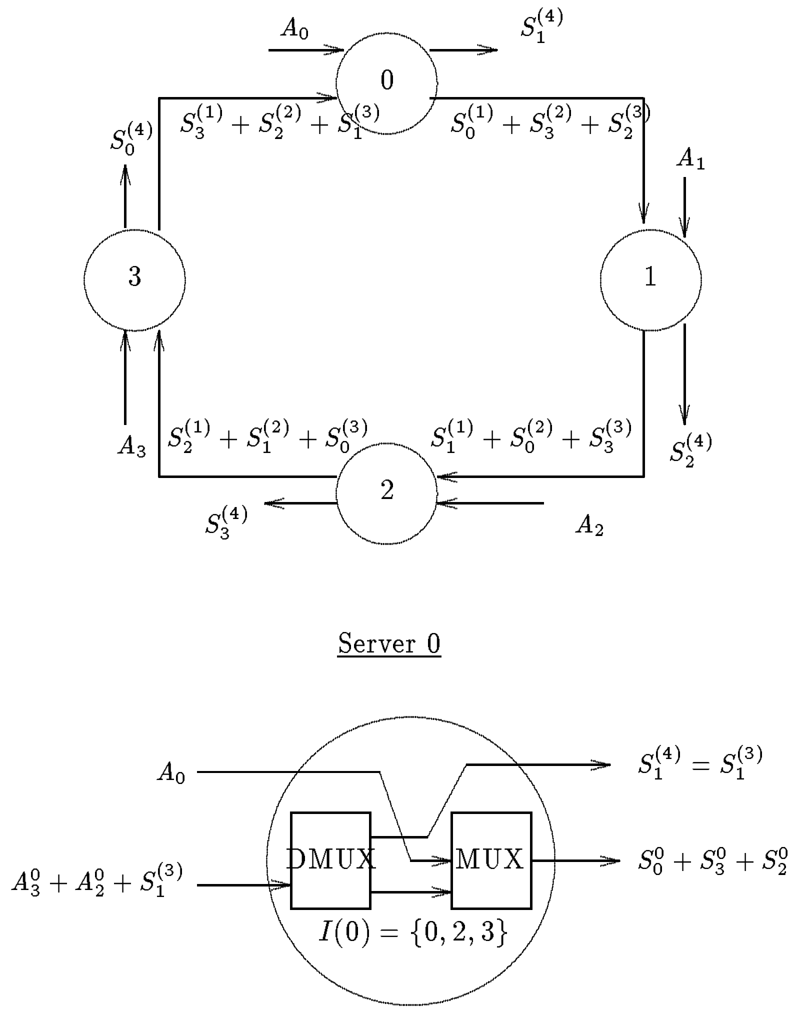


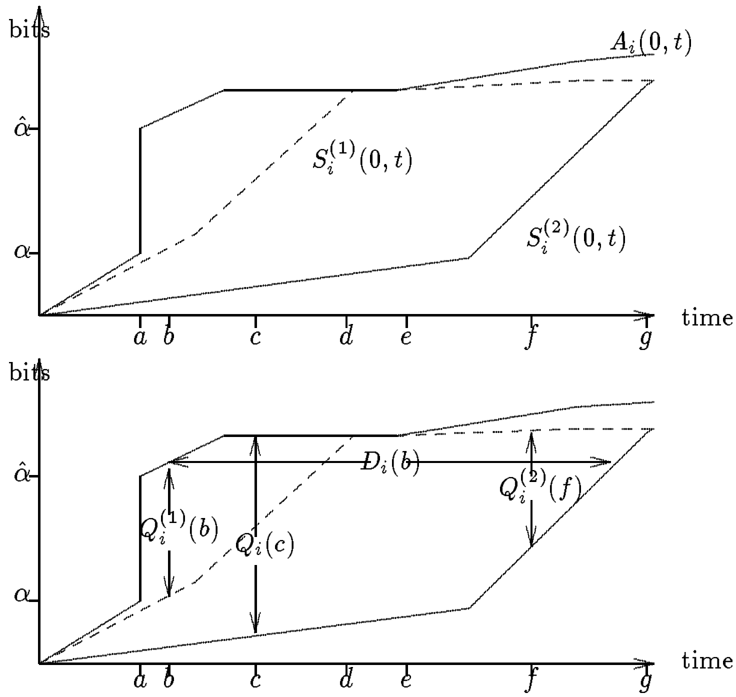
Figure 1: A four server network. The demultiplexer works instantaneously.

$k = P(i, j)$ for a particular session, i, the functions $S_i^{(j)}$ and S_i^k are identical.

3 Network Delay, Backlog and Stability

In this section we extend the notions of session i delay and backlog introduced in [10] to the multiple node case. Given a set of arrival functions for every session in the network, define $Q_i^{(k)}(t)$ to be the session i backlog at node $P(i,k)$ at time t. Similarly, let $Q_i^m(t)$ be the session i backlog at node $m \in P(i)$. Thus, if $m = P(i,k)$, then

$$Q_i^{(k)}(t) = Q_i^m(t) = A_i^m(0,t) - S_i^m(0,t). \tag{5}$$



The first figure shows how session i traffic progresses through the nodes of its route. Notice that the arrival function to node 2 is the session i service function of node 1.

The second figure shows how the backlog and delay can be measured and illustrates the definitions of Section 3.

Figure 2: An Example of Session i flow when $K_i = 2$.

Define the total session i backlog at time t to be

$$Q_i(t) = \sum_{k=1}^{K_i} Q_i^{(k)}(t). \tag{6}$$

Thus, $Q_i(t)$ is the amount of session i traffic buffered in the network at time t. By assumption,

$$Q_i(t) = 0, \quad \forall t \leq 0$$

for every session i. Also, let $D_i(t)$ be the time spent in the network by a session i bit that arrives at time t. Figure 2 shows how to represent the notions of backlog and delay graphically. We see that $D_i(\tau)$ is the horizontal distance between the curves $A_i(0,t)$ and $S_i^{(K_i)}(0,t)$ at the ordinate value of $A_i(0,\tau)$. Clearly, $D_i(\tau)$ depends on the arrival functions $A_1, ..., A_N$, where N is the total number of sessions in the network. We are interested in computing the maximum delay over all time, and over all arrival functions that are consistent with (3) for $i = 1, 2, ..., N$. Let D_i^* be the maximum delay for session i. Then

$$D_i^* = \max_{(A_1,\ldots,A_N)} \max_{\tau \geq 0} D_i(\tau).$$

The backlogs at every node in $P(i)$ can be determined from Figure 2 as shown. Define the maximum backlog for session i, Q_i^*:

$$Q_i^* = \max_{(A_1,\ldots,A_N)} \max_{\tau \geq 0} Q_i(\tau).$$

Note that A_i contains an impulse at time a; As in [10], we adopt the convention that the arrival functions are continuous from the left, so that $A_i(0, a) = \alpha$ and $A_i(0, a^+) = \hat{\alpha}$.

Define the utilization of server m to be

$$u^m = \frac{\sum_{j \in I(m)} \rho_j}{r^m}. \tag{7}$$

A network is defined to be *stable* if $D_i^* < \infty$ for all sessions i. In most of our analysis we will show stability under the assumption that $u^m < 1$ at every server m. Allowing utilizations of greater than 1 would permit backlogs and delays to build up unboundedly, and we have shown elsewhere ([9]) that permitting $u^m = 1$ at each server m can result in problems as well.

The minimum session i backlog clearing rate along its route is

$$g_i = \min_{m \in P(i)} g_i^m. \tag{8}$$

When $g_i > \rho_i$ we define session i to be *locally stable*. Note that if $\phi_i^m = \rho_i$ for all i and $m \in P(i)$ then each session i is locally stable.

Finally, the definitions of system and session busy periods given in [10] for a single node are extended to the multiple node case. A network system (session i) busy period is defined to be a maximal interval B (B_i) such that for every $\tau \in B$ ($\tau \in B_i$), there is at least one server in the network that is in a system (session i) busy period at time τ.

4 Bounds for Locally Stable Sessions

While every route in a data network is acyclic, the union of several routes may result in cycles being induced in the network topology. The presence of these cycles can complicate the analysis of delay considerably, but more importantly, it can lead to feedback effects that drive the system towards instability. This phenomenon has been noticed by researchers from fields as diverse as manufacturing systems [11, 8], communication systems [4] and VLSI circuit simulation [6]. Consider the four node example in Figure 1 (which is identical to Example 2 of Cruz [4]). Suppose the service discipline is FCFS. As an illustration of virtual feedback, notice that $S_0^{(1)}$ depends on the traffic from sessions $2, 3, ..., K-1$, but the form of this traffic is not independent of $S_0^{(1)}$.

In this section we will show that under the GPS service discipline these virtual feedback effects are completely absent for a locally stable session, i, even when the other sessions are not leaky bucket constrained. For notational convenience let $P(i) = (1, 2, ..., K_i)$. The following useful Lemma is straightforward and stated without proof—to see that it is true, recall that we are ignoring propagation delays:

Lemma 1 *For every interval $[\tau, t]$ that is contained in a single session i network busy period:*

$$S_i^{(K_i)}(\tau, t) \geq g_i \ (t - \tau).$$

The Lemma leads us to the main result of this section:

Theorem 1 *If $g_i \geq \rho_i$ for session i:*

$$Q_i^* \leq \sigma_i,$$

$$D_i^* \leq \frac{\sigma_i}{g_i}.$$

Proof. Suppose Q_i^* is achieved at time t, and let τ be the first time before t when there are no session i bits backlogged in the network. Then by Lemma 1, $S_i^{(K_i)}(\tau, t) \geq \rho_i(t - \tau)$. Consequently,

$$Q_i^* \leq (\sigma_i + \rho_i(t - \tau)) - \rho_i(t - \tau) = \sigma_i.$$

An arriving session i bit will be served after at most Q_i^* session i bits have been served. Using Lemma 1 again, these backlogged bits are served at a rate of at least g_i. Therefore:

$$D_i^* \leq \frac{Q_i^*}{g_i} \leq \frac{\sigma_i}{g_i}.$$

□

Note that the delay bound in Theorem 1 is independent of the topology of the network and also of K_i, the number of links in the route taken by the session. Also, it is independent of the σ_j, $j \neq i$. The naive bound on delay arrived at by adding the worst-case delays at each node is $D_i^* \leq \sigma_i \sum_{m=1}^{K_i} \frac{1}{g_i^m}$, illustrating the fact that much better bounds result from analyzing the session i route as a whole. Also, given a locally stable session i, the result of

Theorem 1 is valid for **any** GPS assignment for the other sessions. In fact, the other sessions need not be leaky bucket constrained, nor need the system be stable. An architecture that exploits these features is presented in [2].

When all of the sessions are leaky bucket constrained and $\phi_i^m = \rho_i$ at all m and $i \in I(m)$:

$$Q_i^* \le \sigma_i, \tag{9}$$

and

$$D_i^* \le \frac{\sigma_i}{\rho_i}. \tag{10}$$

5 The Importance of Sessions that are not Locally Stable

When **all** of the sessions are leaky bucket constrained, it is possible to guarantee finite delay even for the sessions that are not locally stable. This is because GPS is work conserving and the token arrival rates are assigned such that $\sum_{j \in I(m)} \rho_j < r^m$ at all nodes m. Thus we may allow g_i to be less than ρ_i for sessions that are not delay sensitive, and much greater than ρ_i for delay sensitive sessions. The merit of this approach has already been discussed in [10] (recall Figure 2 of that paper). Briefly, non-bursty delay sensitive sessions may be given large values of g_i (without signifcantly impacting the performance afforded to the other sessions). Thus, the rest of this paper permits assignments in which some of the sessions may not be locally stable.

As will be apparent shortly, providing such a general analysis is not without its complications. These complications are introduced primarily because of the effects of virtual feedback on performance that were discussed in Section 4. However, despite these difficulties, understanding how to provide good bounds on delay and buffer requirements for a broad class of GPS assignments is useful in determining the range of behavior manifested by this highly flexible service policy. This also yields insight into the range of real-time performance requirements that can be supported. Finally, an important benefit of this line of analysis of GPS networks is that it allows us to relate the results to PGPS networks with variable packet sizes.

6 The All-Greedy Bound for a single node

The presence of sessions that are not locally stable complicates our analysis considerably; yet after performing the analysis we will see that the computation of per-session delay and backlog remains intuitive and quite efficient. There are two steps to providing worst case bounds on delay and backlog: The first consists of characterizing the internal traffic of the network so that at each node, m and $j \in I(m)$ we have σ_j^m such that $A_j^m \sim (\sigma_j^m, \rho_j)$. In the second step, the internal characterization is used to analyze the session i route for delay and backlog.

Following Cruz [3], we calculate *upper bounds* on the minimum value $\sigma_i^{m,out}$ such that $S_i^m \sim (\sigma_i^{m,out}, \rho_i)$. Central to our analytical technique is the concept of the all-greedy bound: These upper bounds will be shown to be quite good for a wide variety of networks. Consider a particular node m. Suppose that for every $j \in I(m)$, we are given that $A_j^m \sim (\sigma_j^m, \rho_j)$. In [10] it was shown that the worst-case delay and backlog for session i (at node m) are each achieved when all the sessions $j \in I(m)$ are simultaneously greedy from time zero, the beginning of a system busy period. However, if two sessions j and p are both served by the same node, n, just before they contend for node m, then it may not be possible for both of them to be simultaneously greedy, as is required in the all-greedy regime. Thus, the acheivable worst-case delay and backlog at node m may be less (but never more) than that calculated under the all-greedy regime.

In the rest of this paper we will make frequent use of the all-greedy bound, in order to simplify procedures for estimating D_i^* and Q_i^*. The following notation is useful in this regard:

We are given σ_j^m, ρ_j for each $j \in I(m)$, such that $\sum_{j \in I(m)} \rho_j < r^m$. Consider a fictitious system in which no traffic enters node m before time zero, and all the sessions at m are greedy starting at time zero. Denote $\hat{A}_i^m$ as the resulting session i arrival function for all $i \in I(m)$. Also denote $\hat{S}_i^m$ as the service function at node m. Recall from [10], that for $t > 0$, as long as $Q_i^m(t) > 0$, the function $\hat{S}_i^m(0,t)$ is piecewise linear and convex-$\cup$ in t. By using the techniques of [10] we can find the smallest value $\hat{\sigma}_i^{m,out}$ such that $\hat{S}_i^m \sim (\hat{\sigma}_i^{m,out}, \rho_i)$. From the discussion above,

$$\hat{\sigma}_i^{m,out} \geq \sigma_i^{m,out}. \tag{11}$$

Thus, we may bound the burstiness of S_i^m by $\hat{\sigma}_i^{m,out}$.

7 Non-Acyclic GPS networks under Consistent Relative Session Treatment

In Section 4 we introduced the notion of virtual feedback, which complicates the analysis of non-acyclic networks, and that can even drive the system into instability. When not all of the sessions are locally stable, it is important to avoid assignments of the ϕ_i's for which reasonable delay guarantees cannot be made. Several examples of the effects of virtual feedback are given in Section 3.3 of [9] that illustrate the difficulty of dealing with this phenomenon. These examples suggest that one of the major causes of poor performance is that a given session is treated poorly relative to a set of sessions at a particular node, but is treated as well relative to the same set of sessions at other nodes. In this section we show that for GPS assignments in which each session is treated "consistently" well relative to the other sessions, the network is stable (as long as $u^m < 1$ at each node m), and tight bounds on delay can be derived.

We begin by the following useful definition:

Definition. *Session j is said to impede a session i at a node m if*

$$\frac{\phi_i^m}{\phi_j^m} < \frac{\rho_i}{\rho_j}.$$

Note that for any two sessions, i and j, that contend for a node m, either session i impedes session j or vice-verse, unless $\frac{\phi_i^m}{\phi_j^m} = \frac{\rho_i}{\rho_j}$, in which case neither session impedes the other.

A Consistent Relative Session Treatment GPS assignment (CRST) is one for which there exists a strict ordering of the sessions such that for any two sessions i, j, if session i is less than session j in the ordering, then session i does not impede session j at any node of the network.

The class of assignments that are CRST is quite broad: For example, consider the special case of a CRST system for which

$$\phi_{ij} = \frac{\phi_i^m}{\phi_j^m}, \quad \forall m \ \text{ s.t. } i, j \in I(m). \tag{12}$$

Thus, whenever sessions i and j contend for service at a link, they are given the same relative treatment. Note that $\phi_{ij} = \frac{\phi_{ip}}{\phi_{jp}}$, where session p is in $P(i) \cap P(j)$. Such CRST systems are called Uniform Relative Session Treatment (URST) systems. Note the following special cases of URST systems:

- For every session i, and node m that is on the session i route: $\phi_i = \phi_i^m$.
- Suppose $\phi_i = \rho_i$ for every session i. Then from (8) each session is locally stable. We call this special case of a URST system, Rate Proportional Processor Sharing (RPPS). Recall the results of Section 4.

We will show that a CRST system is stable if $u^m < 1$ at each node, and will also provide an algorithm for characterizing the internal traffic for every session in a CRST system.

The sessions of any network with a CRST assignment can be partitioned into non-empty classes $H_1, ..., H_L$, such that the sessions in H_k are impeded only by those in H_l, $l < k$. If two sessions i, j, are in the same class their routes are either edge disjoint or

$$\frac{\phi_i^m}{\phi_j^m} = \frac{\rho_i}{\rho_j}$$

at every node, m, that is common to the routes of sessions i and j. Clearly, the sessions in H_1 are not impeded by *any* other session.

Lemma 2 *If $\sum_{j \in I(m)} \rho_j < r^m$, then for any session $j \in H_1$:*

$$\rho_j < \frac{\phi_j^m}{\sum_{p \in I(m)} \phi_p^m} r^m \tag{13}$$

for all nodes $m \in P(j)$.

Proof. Consider a session $j \in H_1$, and suppose that its route includes the node m. Since $\sum_{j \in I(m)} \rho_j < r^m$, there must exist at least one session i, such that

$$\rho_i < \frac{\phi_i^m}{\sum_{p \in I(m)} \phi_p^m} r^m.$$

By definition, i cannot impede session j. Therefore:

$$\frac{\phi_j^m}{\phi_i^m} \geq \frac{\rho_j}{\rho_i} > \frac{\rho_j \sum_{p \in I(m)} \phi_p^m}{\phi_i^m r^m}$$

$$\Rightarrow \phi_j^m r^m > \rho_j \sum_{p \in I(m)} \phi_p^m.$$

Now the claim is proven by rearranging the terms. □

For $j \in H_1$, (13) shows that j's guaranteed backlog clearing rate exceeds ρ_j so that

$$\hat{\sigma}_j^{m,out} = \sigma_j.$$

Using arguments similar to those in Section 4, we have from from (11)

$$\sigma_j^{m,out} \leq \sigma_j. \tag{14}$$

Lemma 2 enables us to upper bound the internal traffic of all the sessions in H_1. The following Lemma will be crucial to us in continuing the process to the sessions belonging to the higher indexed classes:

Lemma 3 *Suppose sessions i and j contend for a link m, and that session j does not impede session i. Then the value of $\hat{\sigma}_i^{m,out}$ is independent of the value of σ_j^m.*

Proof. From Lemma 12 of [10]:

$$\hat{\sigma}_i^{m,out} = Q_i^*. \tag{15}$$

Also, recall that for a single node, Q_i^* is achieved when all of the sessions in $I(m)$ are simultaneously greedy from time zero, the beginning of a session i busy period. Under an all-greedy regime, the service function, $\hat{S}_i^m$ is a continuous piece-wise linear convex-$\cup$ function, with break points corresponding to the times that individual session backlogs clear at node m (see Figure 7 of [10]). The order in which the individual session backlogs clear is shown in [10] to correspond to a *feasible* ordering.

Consider the all-greedy regime obtained when $\sigma_j^m = 0$ (the case $\sigma_j^m > 0$ will follow from this easily), and let the resulting feasible ordering be denoted by $\mathcal{F}$. Let q_i be the (least) time at which maximum backlog is achieved for session i under the all greedy regime, and let e_i^0 be the time that the session i backlog is cleared. Notice that $e_i^0 \geq q_i$. Similarly, let session j terminate its busy period at time e_j^0. We now consider two cases:

<u>Case 1:</u> Session i is less than session j in the feasible ordering, $\mathcal{F}$: Then

$$q_i < e_i^0 < e_j^0.$$

For positive values of σ_j, session i will remain less than session j in the resulting feasible ordering, and the value of e_j^0 can only increase. Thus as σ_j increases from zero, session j remains in a busy period in the interval $[0, q_i]$, and the time q_i remains unchanged, as does the curve $\hat{S}_i^m$ in the interval $[0, e_i^0]$. From (15), it follows that the value of σ_j^m does not influence the value of $\hat{\sigma}_i^{m,out}$.

<u>Case 2:</u> Session i is greater than session j in the feasible ordering, $\mathcal{F}$: Then

$$\hat{S}_j^m(0, e_j^0) = 0 + \rho_j e_j^0 \geq \frac{\rho_i \phi_j^m}{\phi_i^m} e_j^0$$

(the last inequality holds since session j does not impede session i). Thus,

$$\hat{S}_i^m(0, e_j^0) = \frac{\phi_i^m \hat{S}_j^m(0, e_j^0)}{\phi_j^m} \geq \rho_i e_j^0 \tag{16}$$

(the first equality holds from the definiton of GPS), and

$$Q_i^m(e_j^0) \leq \sigma_i.$$

Since $Q_i^* \geq \sigma_i$, and since $Q_i(t)$ strictly increases in the interval $[0, q_i]$ (recall Figure 7 of [10]), it follows that $q_i \leq e_j^0$. Thus, while session j is greater than session i in $\mathcal{F}$, the break-point corresponding to session j in the curve $\hat{S}_i^m$ appears after q_i. Increasing the value of σ_j can only move this break point further out in time, i.e., the time at which the session j busy period terminates can only be greater than e_j^0 for arbitrary values of σ_j^m. Thus, the value of q_i remains unchanged with non-positive values of σ_j. Now by arguments similar to those applied at the end of Case 1, we conclude that the value of σ_j^m does not influence the value of $\hat{\sigma}_i^{m,out}$ in this case as well. □

Lemma 4 *Suppose session i is in $I(m)$ for some node m, and that for every session $j \in I(m)$ that can impede i, σ_j^m is bounded. Then $\sigma_i^{m,out}$ must be bounded as well.*

Proof. From Lemma 3 it follows that $\sigma_i^{m,out}$ can be computed by applying the all-greedy bound to the system in which $\sigma_k = 0$ for all sessions $k \in I(m)$ that do not impede i. Since σ_j^m is bounded for all of the other sessions, i.e. for those sessions that *do* impede i, the resulting value of Q_i^* must be bounded. From (15) the value of $\hat{\sigma}_i^{m,out}$ is bounded, and applying (11) we are done. □

Lemmas 3 and 4 can be used to sequentially characterize the internal traffic of the sessions in classes $H_2, H_3, ..., H_L$. The following procedure specifies the method.

- Compute $H_1, ..., H_L$.
- $k = 1$

  While $k \leq L$, for each session $i \in H_k$

  For $p = 1$ to K_i

  $m = P(i,p)$

  Compute $\hat{\sigma}_i^{m,out}$ given:

  $\sigma_j^m = \hat{\sigma}_j^m$ for all sessions j that impede i at m (computed in earlier steps).

  σ_i^m as computed earlier.

  $\sigma_j^m = 0$ for all sessions j that do not impede i at m.

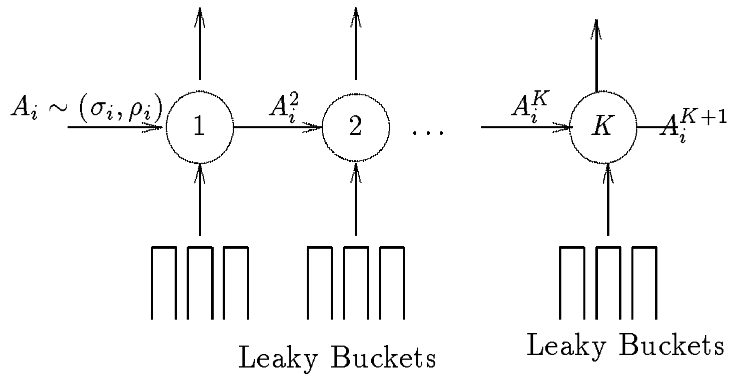
  Set $\sigma_i^{(p)} = \hat{\sigma}_i^{m,out}$.

  $k := k + 1$

Now from (11) we have upper bounds to σ_i^m for every session i and node $m \in P(i)$.

This procedure enables us to show (using an inductive argument on the number of classes) that

Theorem 2 *A CRST GPS network is stable if $u^m < 1$ at each node m.*



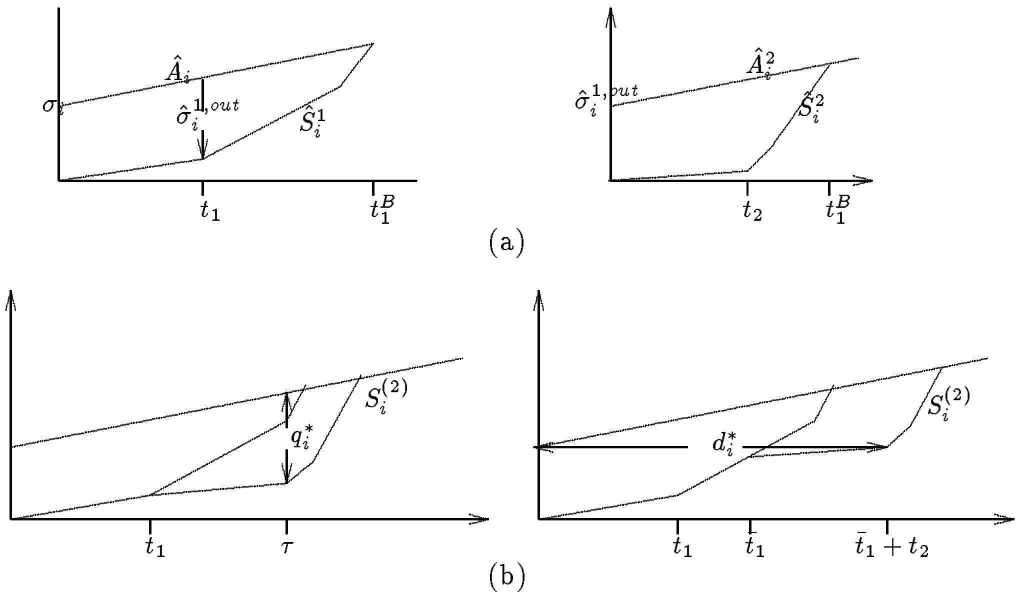
Session i traffic enters the network so that it is consistent with (σ_i, ρ_i), and $A_i^m = S_i^{m-1}$ for $m = 2, 3, ..., K$. The independent sessions at node m are free to send traffic in any manner as long as $A_j^m \sim (\sigma_j^m, \rho_j)$ for every session $j \in I(m) - \{i\}$, $m = 1, 2, ..., K$.

Figure 3: Analyzing the Session i route as a whole, under the Independent Sessions Relaxation.

8 Computing Delay and Backlog for Stable Systems with Known Internal Burstiness

Suppose that we are given a stable GPS system in which the sessions are leaky bucket constrained as in (3), i.e., for every session j and node m such that $j \in I(m)$, we are given a value σ_j^m, such that $A_j^m \sim (\sigma_j^m, \rho_j)$. As we discussed in Section 6, worst case delay (backlog) at a single node of the network can be upper bounded by applying the techniques of [10] when the traffic characterization of sessions sharing that node is known. Under the Additive Method due to [4], we add the worst case bounds on delay (backlog) for session i at each of the nodes $m \in P(i)$ considered in isolation. While this approach works for any server discipline for which the single node can be analyzed, it may yield very loose bounds. For example, when applied to an RPPS system (defined in Section 7) we get $D_i^* \leq K_i \frac{\sigma_i}{\rho_i}$, rather than $D_i^* \leq \frac{\sigma_i}{\rho_i}$. The problem, of course, is that we are ignoring strong dependencies among the queueing systems at the nodes in $P(i)$. In order to improve the bounds the session route is treated as a whole. For notational simplicity we focus on a particular session, i, that follows the route $1, 2, ..., K$. Figure 3 illustrates the system to be analyzed. We will assume that:

1. The sessions $j \in I(m) - \{i\}$ (for $m = 1, 2, ..., K$) are free to send traffic in any manner as long as $A_j^m \sim (\sigma_j^m, \rho_j)$. Thus it is appropriate to call the sessions in $I(m) - \{i\}$, the *independent sessions* at node m $(m = 1, 2, .., K)$.



The curves $\hat{S}_i^1$ and $\hat{S}_i^2$ are shown in (a). Note that $\sigma_i^2 = \hat{\sigma}_i^{1,out}$, and so $\hat{S}_i^1$ and $\hat{S}_i^2$ cannot be determined independently.
Figure (b) shows two staggered greedy regimes. In the first, the sessions in $I(2) - \{i\}$ become greedy at time t_1, which yields a maximum backlog of q_i^* at time τ. In the second staggered greedy regime, the sessions at $I(2) - \{i\}$ wait until time $\bar{t}_1$ to become greedy—this results in a maximum delay of d_i^* for session i at time zero.

Figure 4: Two Staggered Greedy Regimes when $P(i) = \{1, 2\}$

2. Session i traffic is constrained to flow along its route so that

$$A_i^m = S_i^{m-1} \quad m = 2, 3, ..., K.$$

Assumptions 1 and 2 are collectively known as the <u>independent sessions relaxation</u>. This is because while the network topology may preclude certain arrival functions of A_j^k that are consistent with (σ_j^k, ρ_j), these functions are included under the independent sessions relaxation. On the other hand, every arrival function allowable in the network, is allowed under the independent sessions relaxation. Thus, the values of D_i^* and Q_i^* that hold under the independent sessions relaxation, must be upper bounds on the true values of these quantities. The use of all-greedy bounds enables us to compute D_i^* and Q_i^* <u>exactly</u> under the independence relaxation.

In view of our results for the single node case, it would be satisfying if maximum delay (and backlog) were achieved when all the sessions of the network are greedy starting at time zero (the beginning of a system busy period). However, this is generally not true. It turns out that what is required is that the sessions at a particular node j become greedy simultaneously, but only after the sessions at node $j-1$ become greedy. We call this pattern of arrivals a staggered greedy regime. The instants of time at which the sessions become greedy depend on the session for which maximum delay and backlog is being estimated. We will also find that Q_i^* and D_i^* may not both be achieved for the *same* staggered greedy regime. This important point is illustrated in Figure 4.

The possibility of D_i^* and Q_i^* being achieved under different staggered greedy regimes is discouraging from a practical standpoint, especially if computing either one of these quantities involves solving a complicated optimization problem. It would be much more desirable to have a single function from which both delay and backlog can be bounded. (In the single-node case this curve is just $\hat{S}_i$, i.e. Lemma 10 of [10].)

In Section 8.1 we describe such a function, which we call the session i universal curve, $U_i(t)$. This curve is constructed without computing any staggered greedy regimes, and both D_i^* and Q_i^* can be determined efficiently and exactly from it (under the independent sessions relaxation). In addition, the staggered greedy regimes that achieve these worst-case values can also be efficiently determined from $U_i(t)$. In Section 8.2 we prove that these worst-case staggered greedy regimes achieve the same bounds on D_i^* and Q_i^*, as computed from $U_i(t)$.

8.1 The Session i Universal Service Curve

The universal service curve forms the basis for most of the major results in the remainder of this paper. It is a easily constructed by applying the all-greedy bound at each of the hops of the session i route, and allows for a straightforward determination of worst-case delay and backlog. The exact relationship between the session i universal service and the session i service curve is given by the inequality of Lemma 6, but intuitively, the value of the universal service curve at time t, yields a tight bound on the maximum number of session i bits that can ever traverse the network in the first t time units of a network session i busy period.

For notational simplicity, we will focus on a session i such that $P(i) = (1, 2, ..., K)$. The functions $\hat{S}_i^1, ..., \hat{S}_i^K$ can be computed using the internal traffic characterization of Section 7 by using the independent sessions relaxation. Recall that for each node $m = 1, 2, ..., K$, $\hat{S}_i^m$ is continuous, piece-wise linear and is convex-$\cup$ in the range $[0, t_m^B]$, where t_m^B is the duration of the session i busy period at m under the all-greedy regime. Also $\hat{S}_i^m(0) = 0$. Thus it can be specified (in the range $[0, t_m^B]$) by a list of pairs:

$$(s_1^m, d_1^m), (s_2^m, d_2^m), \ldots, (s_{n_m}^m, d_{n_m}^m),$$

where s_j^m is the slope of the j^{th} line segment, d_j^m is its duration and n_m is the number of line segments. Here

$$s_1^m < s_2^m < \ldots < s_{n_m}^m, \tag{17}$$

and

$$\sum_{j=1}^{n_m} d_j^m = t_m^B. \tag{18}$$

We first describe how to construct U_i from $\hat{S}_i^1, ..., \hat{S}_i^K$, and then define the curve analytically. Finally, we establish the relationship between U_i and the session i departures from the network, $S_i^{(K)}$:

Let E_i^k be the collection of all the pairs (s_j^m, d_j^m) for $m = 1, 2, ..., k$—i.e.

$$E_i^k = \bigcup_{m=1}^{k} \bigcup_{j=1}^{n_m} \{(s_j^m, d_j^m)\}.$$

The session i universal service curve, U_i is defined as:

$$U_i(t) = \min\{G_i^K(t), \hat{A}_i(0,t)\},$$

where the curve G_i^k (for $k = 1, 2, ..., K$) is a continuous curve constructed from the elements of E_i^k as follows:

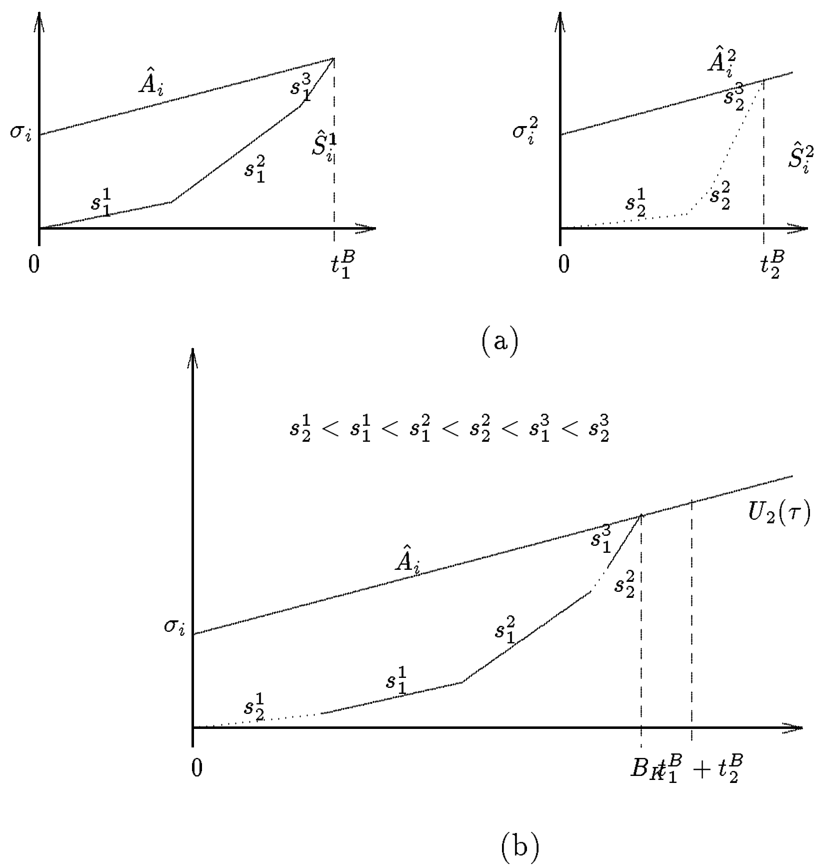
1. Set $G_i^k(0) = 0$, Remaining-in-E $= E_i^k$; $Glist = \phi$; $u = 0$; $t = 0$.
2. Order the elements of E_i^k in increasing order of slope. Remove from Remaining-in-E an element of smallest slope: $e^{new} = (s^{new}, d^{new})$.
   Append Glist with e^{new}. If Remaining-in-E is not empty then repeat step 2.
3. G_i^k is a piece-wise linear convex-$\cup$ function defined in the range $[0, \sum_{m=1}^k t_m^B]$ by the elements of $Glist$[1].
   For $t \geq \sum_{m=1}^k t_m^B$ set

$$G_i^k(t) = G_i^k(\sum_{m=1}^{k} t_m^B) + \hat{A}_i(\sum_{m=1}^{k} t_m^B, t). \tag{19}$$

Figure 8.1 illustrates the construction of U_i for a simple two node example. Note that:

- G_i^k is defined for $k = 1, 2, ..., K$, but U_i is defined in terms of G_i^K.
- For each m, the relative order of the elements from $\hat{S}_i^m$ is preserved in $Glist$.

[1] In the same manner as $\hat{S}_i^m$ was specified earlier.



The two service curves in (a) show $\hat{S}_i^1$ and $\hat{S}_i^2$. In (b) the line segments that make up these curves are concatenated to make a piece-wise linear convex curve that meets $\hat{A}_i$ at time B_K. Thus

$$U_i(t) = \begin{cases} G_i^K(t) & t \leq B_K \\ \hat{A}_i(0,t) & t > B_K. \end{cases}$$

Note that the line segment with slope s_2^3 is never used in the construction of U_2, i.e. $T_2 < t_1^B + t_2^B$.

Figure 5: An example of how U_i is constructed for $K = 2$.

- We still have to show that for any network, the curve G_i^K always meets $\hat{A}_i$—this is established in Lemma 5.

Describing the construction of G_i^K is useful in understanding its form, but we need an analytical definition of the curve in order to prove things about it. The following is a useful, notationally compact definition for times t in the range $[0, \sum_{m=1}^{k} t_m^B]$:

$$G_i^k(t) = \begin{cases} \hat{S}_i^1(0,t), & \text{for } k = 1 \\ \min_{\tau \in [0,t]}\{G_i^{k-1}(\tau) + \hat{S}_i^k(0, t-\tau)\}, \quad t - \tau \leq t_k^B, & \text{for } k \geq 2. \end{cases} \tag{20}$$

To see how (20) corresponds to the algorithm given earlier, expand the recursion in terms of $\tau_1, ..., \tau_k$, where τ_m, corresponds to the minimizing value for node m. Clearly, $\tau_1 = 0$, and define $\tau_{k+1} = t$. Then $\tau_{m+1} - \tau_m \leq t_m^B$ for each $m = 1, 2, .., k$ and

$$\begin{aligned} G_i^k(t) &= \min_{\tau_k \in [0,t]} \min_{\tau_{k-1} \in [0,\tau_k]} \cdots \min_{\tau_2 \in [0,\tau_3]} \left\{ \sum_{m=1}^{k} \hat{S}_i^m(0, \tau_{m+1} - \tau_m) \right\} \\ &= \min_{0 \leq \tau_2 \leq \tau_3 \leq \cdots \tau_k \leq t} \sum_{m=1}^{k} \hat{S}_i^m(0, \tau_{m+1} - \tau_m). \end{aligned} \tag{21}$$

For each m, the quantity $\tau_{m+1} - \tau_m$ corresponds to the total duration of the elements picked for *Glist* from the list describing $\hat{S}_i^m$. Suppose we are given $G_i^1 = \hat{S}_i^1(0,t)$, and wish to compute $G_i^2(t)$ for some $t \in [0, \sum_{m=1}^{2} t_m^B]$. Applying the algorithm to the construction of G_i^2, we determine $\hat{\tau}$, the duration of the elements picked from the list describing $\hat{S}_i^1$. Then $\hat{\tau}$ corresponds to the minimizing value of τ in (20). Thus $G_i^k(t)$ is the curve described by *Glist*. Note that the minimizing values of $\tau_2, ..., \tau_k$ are functions of t.

In the next Lemma we show that $G_i^k(t)$ must meet $\hat{A}_i(0,t)$ at some time before $\sum_{m=1}^{k} t_m^B$:

Lemma 5

$$G_i^k\left(\sum_{m=1}^{k} t_m^B\right) \geq \hat{A}_i\left(0, \sum_{m=1}^{k} t_m^B\right).$$

Proof. Let $\tau_1, ..., \tau_{k+1}$ be the minimizing values of (21).

By definition:

$$\tau_{m+1} - \tau_m \leq t_m^B.$$

for each $m = 1, 2, ..., k$. For $t = \sum_{m=1}^{k} t_m^B$ we must have equality in each of these K inequalities. Thus

$$\hat{S}_i^m(0, \tau_{m+1} - \tau_m) = \hat{S}_i^m(0, t_m^B) = \hat{A}_i(0, t_m^B)$$

(where the second equality follows from the definition of t_m^B), and

$$G_i^k(\sum_{m=1}^{k} t_m^B) = \sum_{m=1}^{k} \hat{S}_i^m(0, t_m^B) = \sum_{m=1}^{k} \hat{A}_i(0, t_m^B) \geq \hat{A}_i(0, \sum_{m=1}^{k} t_m^B).$$

□

Now observe from (19) that for any $t \geq \sum_{m=1}^{k} t_m^B$ we must have:

$$G_i^k(t) \geq \hat{A}_i(0, t). \tag{22}$$

Then there exists $B_k \leq \sum_{m=1}^{k} t_m^B$ such that

$$\begin{aligned} G_i^k(t) &< \hat{A}_i(0,t), \quad t < B_k \\ &= \hat{A}_i(0,t), \quad t = B_k \\ &\geq \hat{A}_i(0,t), \quad t \geq B_k. \end{aligned} \tag{23}$$

Thus,

$$U_i(t) = \begin{cases} G_i^K(t) & t \leq B_K \\ \hat{A}_i(0,t) & t > B_K. \end{cases} \tag{24}$$

Having defined U_i, we now relate it to the session i departures from the network. First, we state two important results that are crucial to the analysis that follows. Lemma 8 establishes that if the independent sessions at a node m are greedy from time zero, then as long as session i remains busy in an interval $[0, \tau]$, the function S_i^m will be identical to $\hat{S}_i^m$ in this interval. Thus session i does not have to be greedy, just busy during the interval. Lemma 9 states that if the independent sessions at a node m are quiet during the interval $[0, \tau]$ and then are greedy starting at τ, then this behavior minimizes $S_i^m(\tau, t)$, the amount of service received by session i at node m from time τ on. The precise statements are in Appendix A and their proofs of follow almost directly from our work in [10]. In the next Lemma we establish the relationship between S_i^m and G_i^m:

Lemma 6 *Consider a given arrival function, A_i, and a given time τ such that $Q_i(\tau) = 0$. Then for each m, $1 \leq m \leq K$, each $t > \tau$:*

$$S_i^m(\tau, t) \geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(t - V)\}. \tag{25}$$

Proof. See Appendix A. □

In the next section we will show that $G_i^m(t)$ is the amount of service given to session i under a specific staggered greedy regime called the (m, t)-staggered greedy regime. Thus Lemma

6 shows that the service to session i is minimized when such a staggered greedy regime is delayed by an appropriate amount, which is the minimizing value of V. Equation (25) facilitates the following bounds on delay and backlog:

Theorem 3 *For every session i:*

$$Q_i^* \leq \max_{\tau \geq 0}\{\hat{A}_i(0,\tau) - G_i^K(\tau)\}, \tag{26}$$

and

$$D_i^* \leq \max_{\tau \geq 0}\left\{\min\{t : G_i^K(t) = \hat{A}_i(0,\tau)\} - \tau\right\}. \tag{27}$$

Proof. See Appendix A. □

The inequalities (26) and (27) illustrate the importance of the universal curve. To find the bound on D_i^* compute the maximum horizontal distance between the curves $A_i(0,t)$ and $U_i(t)$ at the ordinate value of $\hat{A}_i(0,t)$. Similarly, Q_i^* is bounded by the maximum vertical distance between the two curves. In the next section, we will show that these bounds are *achieved* for (K,t)-staggered greedy regimes under the independent sessions relaxation.

8.2 The (K,t)-Staggered Greedy Regime

In this section we make clear the relationship between staggered greedy regimes and the session i universal curve U_i. As in the previous sections, we will focus on staggered greedy regimes with respect to a session i and assume that $P(i) = \{1,2,...,K\}$.

Any staggered greedy regime can be characterized by a vector

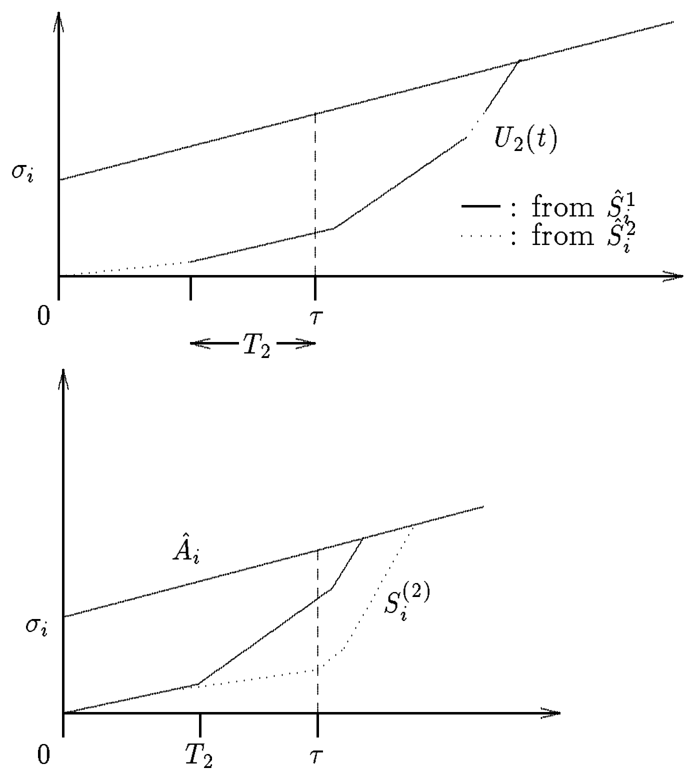
$$(T_1,...,T_K), \quad T_1 \leq T_2 \leq \ldots \leq T_K$$

such that all the sessions at node 1 are simultaneously greedy starting at time T_1, and the independent sessions at node j do not send any traffic in the interval $[T_1,T_j)$, but are simultaneously greedy starting at time T_j. Observe that the first staggered greedy regime in Figure 4(b) can be characterized by $(0,t_1)$ and the second by $(0,\bar{t}_1)$.

A (K,t)-staggered greedy regime, $t \leq B_K$, is the staggered greedy regime characterized by $(0,T_2,....,T_K)$ such that

$$\sum_{k=1}^{K} \hat{S}_i^k(0,T_{k+1}-T_k) = G_i^K(t) \tag{28}$$

where $T_1 = 0$, $T_{K+1} = t$ and $T_{k+1} - T_k \leq t_k^B$ for $k = 1,2,...,K$.



The top figure shows the curve U_2 that was constructed from $\hat{S}_i^1$ and $\hat{S}_i^2$. In order to find the $(2, \tau)$-staggered greedy regime, add the durations of the line segments taken from $\hat{S}_1$ that are in $U_2(t)$, $t \leq \tau$. This sum is T_2, the time that the independent sessions at node 2 become greedy. This characterizes the staggered greedy regime which is shown in the bottom figure.

Figure 6: Computing a (k, t)-Staggered Greedy Regime when $P(i) = \{1, 2\}$

Note that

- Since $t \leq B_K$, $G_i^K(t) = U_i(t)$.
- For each $k = 1, 2, ..., K-1$ the staggered greedy regime defined by $(0, T_2, ..., T_k)$ describes a (k, T_{k+1})-staggered greedy regime.

Comparing (28) with (21) it is clear that $(T_2, ..., T_K)$ is a minimizing vector in (21). Thus, the universal service curve can be used to determine $T_2, ..., T_K$. This is illustrated in Figure 6 for the simple case of $K = 2$. Notice from the figure that in the range $[0, T_2]$, S_i^2 is comprised of the line segments belonging to $\hat{S}_i^1$ that make up the universal curve in the range $[0, \tau]$. Also, notice that in Figure 6

$$S_i^2(0, \tau) = U_i(\tau).$$

It turns out that this is true in general:

Theorem 4 *For any (K, t)-staggered greedy regime:*

$$S_i^{(K)}(0, t) = G_i^K(t).$$

Proof. See Appendix B.□

Figure 7 shows how to construct the staggered greedy regimes that maximize backlog and delay. From Theorems 3 and 4 we have the main theorem of this section:

Theorem 5 *Under the independent sessions relaxation, D_i^* and Q_i^* are each achieved under (K, t)-staggered greedy regimes.*

Now since the values of D_i^* and Q_i^* achieved under the independent sessions relaxation are upper bounds to the actual values of these quantities, we have shown how to find upper bounds on session backlog and delay. Also, since an infinite capacity link can always simulate a finite capacity link, worst case session i backlog and delay calculated under this relaxation must upper bound the values of these quantities for finite capacity links.

9 Propagation Delay

It is easy to incorporate deterministic propagation delays into our network framework: Suppose that every bit transmitted on link (i, j), incurs a delay of $d_{i,j}$ time units. Then each link acts as a constant delay element, and the characterization of internal traffic (using the method of Section 7) remains the same. A natural modification of the independent

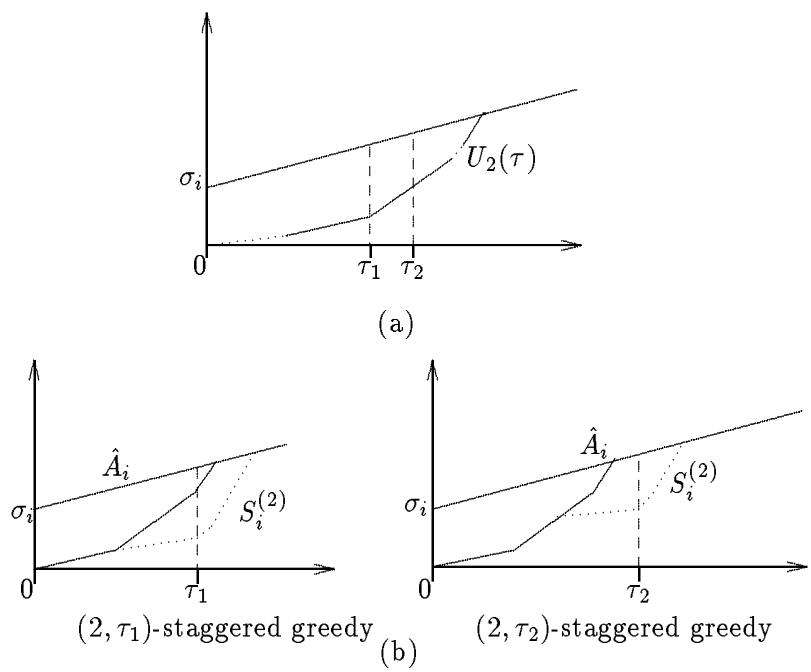


Figure (a) shows the session i universal curve. Notice that for this curve "backlog" is maximized at time τ_1 and "delay" is maximized at time τ_2. Figure (b) shows the two staggered greedy regimes corresponding to these times. Notice that the backlog at time τ_1 in the first regime is exactly equal to the "backlog" at time τ_1 in (a), and similarly the delay at time τ_2 in the second regime is exactly equal to the "delay" at that time in (b).

Figure 7: The staggered greedy regimes that maximize backlog and delay under the independent sessions relaxation.

sessions relaxation allows us to bound end-to-end delay as well: Consider a session i such that $P(i) = 1, 2, ..., K$: Also, let $d_{0,1}$ be the propagation delay on the access link. Then

1. The independent sessions at node m, $j \in I(m) - \{i\}$ (for $m = 1, 2, ..., K$) are free to send traffic in any manner as long as $A_j^m \sim (\sigma_j^m, \rho_j)$.

2. Session i traffic is constrained to flow along its route so that

$$A_i^m(\tau, t) = S_i^{m-1}(\tau - d_{m-1,m}, t - d_{m-1,m}) \qquad m = 2, 3, ..., K.$$

In view of the analysis of Section 8:

$$D_i^* \leq \sum_{m=1}^{K} d_{m-1,m} + D_i^{*,\text{noprop}},$$

where $D_i^{*,\text{noprop}}$ is the worst-case session i delay computed for the same characterization of internal traffic when propagation delays are zero. The number of bits in "flight" on a link (l, m) is at most

$$q_{l,m} = r_l d_{l,m}. \tag{29}$$

Thus

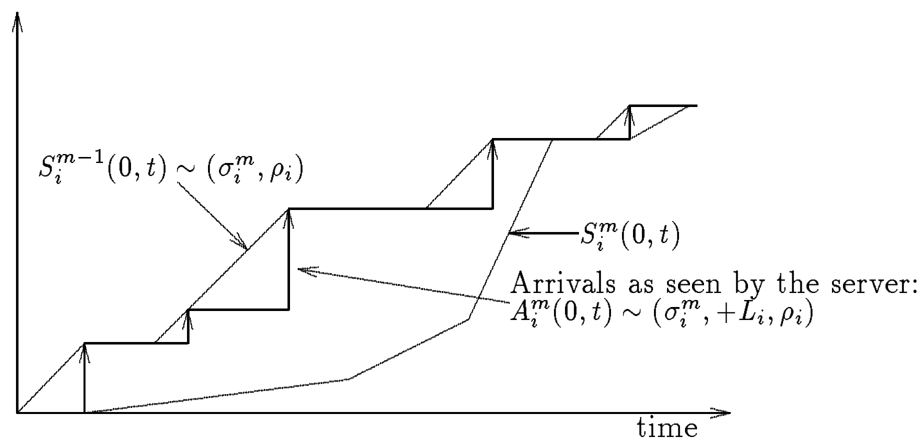
$$Q_i^* \leq \sum_{m=1}^{K} q_{m-1,m} + Q_i^{*,\text{noprop}}.$$

10 PGPS networks

When packet sizes are so small that the maximum packet transmission time at any link of the network is negligible, we may conclude from Theorem 2 of [10], that the behavior of GPS and PGPS are (essentially) identical. Thus in this case, all of the bounds for GPS networks in Sections 7 and 8 can be expected to apply to PGPS networks as well.

In the more general case in which packet sizes are not negligible, there are two effects to consider. First, packets must be served non-preemptively, i.e., once the server has begun serving a packet, it must continue to do so until completion. Second, no packet is eligible for service until its *last* bit has arrived, since in most networks with heterogeneous link speeds, packets are not transmitted until they have *completely arrived.* Stated differently, we assume that service is not virtual cut-through. Thus, if $m - 1$ and m are successive nodes on a session i's route, we cannot assume, as we did in Section 8, that $S_i^{m-1} = A_i^m$. In fact, for $P(i) = \{1, 2, ..., K_i\}$:

$$S_i^{m-1}(\tau, t) + L_i \geq A_i^m(\tau, t) \geq S_i^{m-1}(\tau, t) - L_i, \quad m = 2, ..., K_i, \quad \tau < t. \tag{30}$$



$A_i^m(0,t)$ represents the cumulative arrivals seen by server, m. The length of each impulse of $A_i^m(0,t)$ is bounded by L_i, the maximum packet size for session i. Since $L_i \leq \sigma_i^m$, it can be seen from the figure that $A_i^m \sim (\sigma_i^m + L_i, \rho_i)$.

Figure 8: A system in which the packet sizes are non-negligible.

where $L_i \leq \sigma_i$ is the maximum packet size for each session i. This effect is illustrated in Figure 10. Notice that since the GPS server does not begin serving a packet until its last bit has arrived, it "sees" the arrivals as a series of impulses, such that the height of each impulse is at most L_i. However, since we are not assuming any peak rate constraint in the input characterizations, A_i^m, is consistent with $(\sigma_i^m + L_i, \rho_i)$. These important differences notwithstanding, we will still find the results of the prevous sections to be very useful in the more general case of non-negligible packet sizes. We first enforce the non-virtual cut-through effect, but allow preemptive service, and then incorporate the effects of non-preemptive service.

10.1 Non-Cut-Through GPS

To analyze networks of GPS servers with non-negligible packet sizes we follow the same steps as we did in Sections 7 and 8—we first characterize the internal traffic in terms of leaky bucket parameters, and then bound the worst-case delay and backlog for each session by analyzing its route as a whole. To incorporate the effects of finite packet lengths we stipulate that (30) holds.

Consider a GPS network with CRST assignments. The internal traffic can be characterized using essentially the same procedure as in 7 to compute the all-greedy bounds. To analyze the session i route given internal characterization of the traffic, we proceed as follows: Define $\hat{S}_i^m$ to be the session i output at node m under the all-greedy regime. Then the session i universal service curve is computed as it was in Section 8. Note that Lemma 8 also holds. However, Lemma 6 and Theorem 3 must be modified in order to incorporate (30). The proofs of these modified results follow the arguments made in Appendix A—details are available in Chapter 4 of [9]:

In what follows we assume (for notational simplicity) that $P(i) = \{1, 2, ..., K\}$:

Lemma 7 *Consider some time τ such that $Q_i(\tau) = 0$. Then for each m, $1 \leq m \leq K$, each $t > \tau$:*

$$S_i^m(\tau, t) + L_i \geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(0, t - V)\} - mL_i. \tag{31}$$

Theorem 6 *For every session i:*

$$Q_i^* \leq \max_{\tau \geq 0} \{\hat{A}_i(0, \tau) - G_i^K(\tau)\} + KL_i. \tag{32}$$

Theorem 7 *For every session i, define D_i^* to be the maximum session i packet delay. Then*

$$D_i^* \leq \max_{\tau \geq 0} \left\{ \min\{t : G_i(t) = \hat{A}_i(0, \tau) + (K - 1)L_i\} - \tau \right\}. \tag{33}$$

Theorems 6 and 7 allow us to bound D_i^* and Q_i^* in terms of the universal service curve.

10.2 Non-Preemptive Service: PGPS

Suppose we are given a network of PGPS servers such that the assignments of the ϕ_i's meet the CRST requirements of Section 7. Recall that a CRST assignment ensures a partition of the sessions into classes $H_1, H_2, ...$ such that a session in class c may only be impeded by sessions belonging to classes indexed lower than c. Consider a session $j \in H(1)$ and let $P(j) = \{1, 2, ..., K_j\}$. We know from Corollary 1 of [10] that

$$\hat{Q}_j^1(\tau) - Q_j^1(\tau) \leq L_{\max} \tag{34}$$

for all τ where $\hat{Q}_j^m$, and Q_j^m represent the session i backlogs at node m, under PGPS and GPS respectively. Thus

$$\hat{Q}_j^{1,*} \leq Q_j^{1,*} + L_{\max}.$$

Also, from Lemma 12 of [10]:

$$\sigma_j^{\text{out}} = Q_j^*. \tag{35}$$

Equations (34) and (35) allow one to characterize the internal traffic at each node in $P(j)$ using essentially the same procedure as in Section 7: If applying the all-greedy bound to a node (assuming GPS service) yields a bound of $\sigma_j^{out,m} = \alpha$, then the bound on this quantity under PGPS is just $\alpha + L_{\max}$.

The next step is to analyze delay along the session i route: In Chapter 4 of [9] we prove the following Theorem that allows us to relate worst-case session delay in a PGPS network to the universal service curve, and consequently to GPS networks. The proof is omitted here because of constraints on space.

Theorem 8 *For each session i:*

$$D_i^{*,\mathrm{PGPS}} \leq \max_{\tau \geq 0} \left\{ \min\{t : G_i^K(t) = \hat{A}_i(0,\tau) + (K-1)L_i\} - \tau \right\} + \sum_{m=1}^{K} \frac{L_{\max}}{r^m}. \tag{36}$$

where the universal service curve, G_i^K is computed using the algorithm given in Section 10. section.

Note that the GPS network being considered here has internal characterization *identical* to the PGPS network which will in general have more bursty traffic. It is interesting to observe that as the link speeds become faster, i.e., as $r^m \to \infty$,

$$D_i^{*,\mathrm{PGPS}} = D_i^{*,\mathrm{GPS}}.$$

10.3 Rate Proportional Processor Sharing Networks

In this section we will interpret the results of the previous section for a special CRST assignment. Under RPPS Networks $\phi_i^m = \rho_i$ for every session i and $m \in P(i)$. Recall that in Section 4 we analyzed RPPS networks when the packet sizes are negligible, and derived the bounds (9) and (10) for delay and backlog respectively. Here the corresponding bounds for PGPS service are derived.

Applying the fact that the slope of G_i^K is never less than ρ_i for each session i to (36), we have:

$$D_i^{*,\mathrm{PGPS}} \leq \frac{\sigma_i + 2(K-1)L_i}{\rho_i} + \sum_{m=1}^{K} \frac{L_{\max}}{r^m}. \tag{37}$$

The first term on the RHS is likely to dominate in most instances. In particular, in high speed networks we assume that $r^m \to \infty$, and we have

$$D_i^{*,\mathrm{PGPS}} \leq \frac{\sigma_i + 2(K-1)L_i}{\rho_i}. \tag{38}$$

Also, as $L_{\max} \to 0$, we get (10). The extra delay of $\frac{2(K-1)L_i}{\rho_i}$ in (37) does not diminish with increasing link speed.

This example and (38) strongly indicate that small packet lengths should be chosen in

RPPS networks so that the term $\frac{L_i}{\rho_i}$ is small. For ATM networks, in which the packets are about 400 bits long, this holds for most kinds of applications. Finally note that for a locally stable session i, with minimum backlog clearing rate g_i we have

$$D_i^{*,\mathrm{PGPS}} \leq \frac{\sigma_i + 2(K-1)L_i}{g_i} + \sum_{m=1}^{K} \frac{L_{\max}}{r^m}. \tag{39}$$

even when the other sessions are not leaky bucket constrained. This fact has been used in designing an architecture based on PGPS servers in [2].

11 Conclusions and Extensions

Per-session bounds were derived for the leaky bucket constrained sessions of arbitrary topology GPS and PGPS networks. These bounds are considerably more tight than those derived by treating each hop independently and adding the worst-case delays of each hop. The tightness of the bounds makes it possible to examine the effect of various strategies on per session performance by adjusting the assignments of the GPS servers within a broad range. Thus, our work provides a framework within which the issue of providing performance guarantees to a a wide variety of co-existing session types can be studied. Work along these lines can be found in [2].

An important part of any flow control scheme, and one that is missing from this paper is call-admission. We have not treated the important and dual problem of matching given delay requirements to a set of GPS assignments, which may be difficult, except in the case of locally stable sessions.

Another area for future research is the incorporation of traffic types that require real-time performance but that cannot predict the exact values of their leaky bucket parameters at session set-up time. Since PGPS provides for a natural isolating mechanism, i.e. the per session backlog clearing rate, it may be appropriate for the multiplexing of various classes of traffic, only some of which may be leaky bucket constrained. Recall that bounds given to locally stable sessions still hold under such an arrangement.

Our analysis deals with a static problem in which the real-time nature of call-arrival is not taken into account. There may be many ways in which the algorithms we have given can be extended to incorporate these effects. Although we have not focused on the implementation costs of building PGPS servers or on the computational overhead in computing the bounds, our work can be extended to investigate these issues in more depth.

Finally, the worst-case nature of the analysis and the generality of the leaky bucket arrival constraint may result in bounds that are overly conservative for actual systems.

Thus important areas of future work include probabilistic analyses powerful enough to deal adequately with non-acyclic networks, and that also have the ability to distinguish properly the effects of different work-conserving policies on network performance.

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Appendix A

Lemma 8 *Suppose the independent sessions relaxation holds, and that t is contained in a session i busy period at node m that begins at time 0. Also, suppose that none of the independent sessions have sent any traffic before time 0, and that each is greedy starting at time zero. Then S_i^m is identical to $\hat{S}_i^m$ in the range $[0,t]$.*

Lemma 9 *Suppose the independent sessions relaxation holds, and that time t is contained in a session i busy period at server m that starts at time $\tau \leq t$. Then for all $t \geq \tau$, $S_i^m(\tau,t)$ is minimized over all arrival functions when for every independent session p at node m:*

1. *$A_p^m(0,\tau) = 0$.*

2. *Session p is greedy from time τ.*

Proof of Lemma 6: For $m = 1$, (25) states that

$$S_i^1(\tau,t) \geq \min_{V \in [\tau,t]} \{A_i(\tau,V) + \hat{S}_i^1(0,t-V)\}.$$

Choosing V to be last time in the interval $[\tau,t]$ that session i begins a busy period at node 1:

$$\begin{aligned} S_i^1(\tau,t) &\geq A_i(\tau,V) + \hat{S}_i(0,t-V) \\ &\geq \min_{V \in [\tau,t]} \{A_i(\tau,V) + \hat{S}_i^1(0,t-V)\}. \end{aligned} \tag{40}$$

Now assume the result for nodes $1,2,...,m-1$. Then, letting t_m be the last time in the interval $[\tau,t]$ that session i is in a busy period at node m:

$$S_i^m(\tau,t) = S_i^{m-1}(\tau,t_m) + S_i^m(t_m,t). \tag{41}$$

By the induction hypothesis:

$$S_i^{m-1}(\tau, t_m) \geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V)\}. \tag{42}$$

Also, from Lemma 9:

$$S_i^m(t_m, t) \geq \hat{S}_i^m(0, t - t_m). \tag{43}$$

Substituting (42) and (43) into (41):

$$\begin{aligned}
S_i^m(\tau, t) &\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V)\} + \hat{S}_i^m(0, t - t_m) && (44)\\
&\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^{m-1}(t_m - V) + \hat{S}_i^m(0, t - t_m)\} && (45)\\
&\geq \min_{V \in [\tau, t_m]} \{A_i(\tau, V) + G_i^m(t - V)\} && (46)\\
&\geq \min_{V \in [\tau, t]} \{A_i(\tau, V) + G_i^m(t - V), \} && (47)
\end{aligned}$$

where the inequality in (46) follows from the definition of G_i^m in (20). □

Proof of Theorem 3: We first show (26): For some given set of arrival functions $A_1, ..., A_N$:

$$Q_i(t) = A_i(0, t) - S_i^K(0, t).$$

From Lemma 6,

$$\begin{aligned}
Q_i(t) &\leq A_i(0, t) - \min_{V \in [0, t]} \{A_i(0, V) + G_i^K(t - V)\} && (48)\\
&= A_i(0, t) - A_i(0, V_{\min}) + G_i^K(t - V_{\min}) && (49)
\end{aligned}$$

where $V_{\min}$ is the minimizing value of V. Thus

$$\begin{aligned}
Q_i(t) &\leq A_i(V_{\min}, t) - G_i^K(t - V_{\min}) && (50)\\
&\leq \hat{A}_i(0, t - V_{\min}) - G_i^K(t - V_{\min}) && (51)\\
&\leq \max_{\tau \geq 0} \{\hat{A}_i(0, \tau) - G_i^K(\tau)\}, && (52)
\end{aligned}$$

and (26) follows.

Next we show (27): For a given set of arrival functions, $A_1, ..., A_N$ and $t \geq 0$, we have from Lemma 6:

$$S_i^K(0, t) \geq \min_{V \in [0, t]} \{A_i(0, V) + G_i^K(t - V)\}.$$

Thus, for all $\hat{t} \geq 0$:

$$
\begin{aligned}
D_i(\hat{t}) &= \min\left\{t : S_i^K(0,t) = A_i(0,\hat{t})\right\} - \hat{t} && (53)\\
&\leq \min\left\{t : \min_{V\in[0,t]}\{A_i(0,V) + G_i^K(t-V)\} = A_i(0,\hat{t})\right\} - \hat{t} && (54)\\
&= \min\left\{t : A_i(0,V_{\min}) + G_i^K(t-V_{\min}) = A_i(0,\hat{t})\right\} - \hat{t} && (55)\\
&= \min\left\{t : G_i^K(t-V_{\min}) = A_i(V_{\min},\hat{t})\right\} - \hat{t} && (56)\\
&\leq \min\left\{t : G_i^K(t) = A_i(V_{\min},\hat{t})\right\} + V_{\min} - \hat{t} && (57)\\
&\leq \min\left\{t : G_i^K(t) = \hat{A}_i(0,\hat{t}-V_{\min})\right\} + V_{\min} - \hat{t} && (58)\\
&\leq \min\left\{t : G_i^K(t) = \hat{A}_i(0,\hat{t}-V_{\min})\right\} - (\hat{t}-V_{\min}) && (59)\\
&\leq \max_{\tau\geq 0}\left\{\min\{t : G_i^K(t) = \hat{A}_i(0,\tau)\} - \tau\right\}. && (60)
\end{aligned}
$$

In (55) we choose the *smallest* minimizing value of V. Then $V_{\min} \leq \hat{t}$, since $G_i^K(t-V_{\min}) \geq 0$.
□

Appendix B

The following Lemma establishes (among other things) that for a (K,t)-staggered greedy regime, backlogs are not built up at node m prior to time T_m.

Lemma 10 *Suppose we are given a (K,t)-staggered greedy regime characterized by $(0,T_2,...,T_K)$, $t \leq B_K$, and a node $k \in \{1,2,...,K\}$.*
For each $j = 1,2,...,k-1$, and $\tau \in [T_j, T_{j+1}]$:

$$A_i^k(0,\tau) = \left(\sum_{m=1}^{j-1} \hat{S}_i^m(0,T_{m+1}-T_m)\right) + \hat{S}_i^j(0,\tau-T_j), \tag{61}$$

and for $\tau > T_k$:

$$S_i^k(0,\tau) = \min\{\hat{A}_i(0,\tau), \sum_{m=1}^{k-1} \hat{S}_i^m(0,T_{m+1}-T_m) + \hat{S}_i^k(0,\tau-T_k)\}. \tag{62}$$

Proof. We proceed by induction on k: For $k=1$ only (62) applies. Since $S_i^1 = \hat{S}_i^1$ the basis step is shown. Now assume the result for nodes $1,2,...,k-1$.
Observe by definintion that $(0,T_2,...,T_{k+1})$ is a (k,T_{k+1})-staggered greedy regime. We begin by showing that $Q_i^k(\tau) = 0$ for all $\tau \leq T_k$, i.e. that

$$S_i^k(0,\tau) = S_i^{k-1}(0,\tau), \qquad \text{for all } \tau \leq T_k. \tag{63}$$

The equation (61) follows directly from (63) Suppose (63) is false. Then $Q_i^k(\tau) > 0$ for some $\tau \leq T_k$. Since the independent sessions at k are quiet during the interval $[0, T_k]$ it follows that there is at least one interval before T_k during which $S_i^{(k-1)}$ has slope greater than r_k (where r_k is the rate of k). Since the slope of $\hat{S}_i^k(0,t)$ is never greater than r_k for $t \in [0, t_k^B]$. and $T_1, T_2, ..., T_k$ are derived from the the minimization of (21), it follows that $T_{k+1} - T_k = t_k^B$. We have shown in [10] that no node k busy period can be longer than t_k^B time units, so it follows that $Q_i^k(T_{k+1}) = 0$. Thus

$$S_i^k(0, T_{k+1}) = \hat{A}_i(0, T_{k+1}) = G_i^k(T_{k+1}) - Q_i^k(T_k),$$

where the first equality is from the induction hypothesis and the second equality follows directly from the definition of G_i^k. Then $G_i^k(T_{k+1}) \geq \hat{A}_i(0, T_{k+1})$, and

$$T_{k+1} = B_k.$$

Now, let $[a, a+\Delta]$, such that $\Delta > 0$ and $a + \Delta \leq T_k$, be an interval during which $S_i^{(k-1)}$ has largest slope, and such that this slope belong to a single node, $j < k$. As we have already argued, the slope of $S_i^{(k-1)}$ during this interval must be greater than r_k, since $Q_i^k(\tau) > 0$ for some $\tau < T_k$. Then the staggered greedy regime characterized by

$$\hat{T} = (0, T_2, ..., T_j - \Delta, T_{j+1} - \Delta, ..., T_k - \Delta)$$

is a $(k, T_{k+1} - \Delta)$-staggered greedy regime. I.e.,

$$\sum_{m=1}^{k} \hat{S}_i^m(0, \hat{T}_{m+1} - \hat{T}_m) = G_i^k(t - \Delta), \tag{64}$$

where $\hat{T}_{k+1} = T_{k+1} - \Delta$. Now since $\hat{T}_{k+1} - \hat{T}_k = t_k^B$, it follows from similar reasoning as above that under $\hat{T}$, session i is not backlogged at k at time $\hat{T}_{k+1}$, and that therefore

$$\hat{T}_{k+1} = B_k.$$

Thus

$$\hat{T}_{k+1} = T_{k+1} - \Delta = B_k.$$

But this implies that $\Delta = 0$, which is a contradiction and so (63) holds.

We are now left to show (62). Since $Q_i^k(T_k) = 0$, it is sufficient to establish that $\hat{S}_i^{k-1}(T_k, \tau) \leq \hat{S}_i^k(0, \tau - T_k)$ for all $\tau \in [T_k, T_{k+1}]$. It is straightforward to argue that this must be true from the minimization of (21). □

To show Theorem 4, pick $\tau = t > T_K$ in Lemma 10. Then (62) applies, and since $t \leq B_K$ the result follows.

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