UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Midterm 1 Solutions
Spring 2011

Problem 1 [15] Answer the following questions briefly but clearly. **You must justify your T/F answer.**

(a) True/False: If X and Y are random variables then $\mathrm{var(X+Y)} \geq \max\{\mathrm{var(X)}, \mathrm{var(Y)}\}$.

Solution: False. For example, take your favorite random variable X with $\mathrm{var}(X) > 0$ and set $Y = -X$. Then $\mathrm{var}(X+Y) = \mathrm{var}(0) = 0$, but $\mathrm{var}(X) > 0$.

(b) True/False: Let X be a normal random variable with mean a^2 and variance b^4, and let $Y = a^2 X + b^3$. Then the random variable $Z = \frac{1}{10}\frac{Y - a^4 - b^3}{a^2 b^2}$ is a normal random random variable with mean zero and variance .01.

Solution: True. Let $N \sim \mathcal{N}(0,1)$, then we can write $X = b^2 N + a^2$. Hence $Y = a^2 b^2 N + a^4 + b^3$, and indeed $Z = 0.1N$ is a normal r.v. with zero mean and variance 0.01.

(c) True/False: If X and Y are independent exponential random variables with parameters λ_x and λ_y respectively, then $\min\{X, Y\}$ is an exponential random variable with parameter $\min\{\lambda_x, \lambda_y\}$

Solution: False. We know that $\mathbf{P}(X > t) = e^{-\lambda_1 t}$ and similarly for Y. Now

$$\mathbf{P}(\min(X,Y) \geq t) = \mathbf{P}(X \geq t)\mathbf{P}(Y \geq t) = e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)t},$$

so $\min(X,Y)$ is an exponential r.v. with parameter $\lambda_1 + \lambda_2$.

(d) True/False: If two events are dependent they cannot be conditionally independent.

Solution: False. Let p be uniformly distributed on $[0,1]$ and let X_1, X_2 be two independent coin flips with parameter p. If we don't know p then clearly X_1, X_2 are not independently, but condioned on p the flips are independent.

(e) True/False: For any continuous random variable that has finite expectation, $E[X]$: $P(X \leq E[X]) = 0.5$.

Solution: False. For example, consider the random variable X with pdf

$$f_X(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x \leq 2 \\ \frac{1}{8} & \text{if } -4 \leq x \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Then $\mathrm{E}\, X = -\frac{1}{2}$ but $\mathbf{P}(X \leq -\frac{1}{2}) = \frac{7}{16} < \frac{1}{2}$.

Problem 2[50] Answer the following questions briefly but clearly.

(a) A couple has two children. The probability that they will conceive a boy =0.5 and births are independent of each other. If we are told that their second born is a boy, what is the probability that the first born is a boy?

Solution: The two events are independent, so the probability that the first born is a boy is still 1/2. Note that had we not included the birth order the answer would have been different: if you know there is one boy, and you ask what is the probability the other child is a boy, you get 1/3 since the possible combinations are BB, BG, or GB, each having probability 1/3.

(b) Three persons each rolls a fair 6-sided die once. Let A_{ij} be the event that person i and person j roll the same face. Are the events A_{12}, A_{13}, and A_{23} jointly independent? Are they pairwise independent?

Solution: The events are pairwise independent but not jointly independent. It is not hard to check that $\mathbf{P}(A_{13}|A_{12}) = 1/6 = \mathbf{P}(A_{13}) = \mathbf{P}(A_{12})$, which by the definition of conditional probability shows that $\mathbf{P}(A_{13} \cap A_{12}) = \mathbf{P}(A_{13})\mathbf{P}(A_{12})$. One can argue similarly for the other two pairs, showing pairwise independence. However, $\mathbf{P}(A_{12} \cap A_{13} \cap A_{23}) = \mathbf{P}(A_{13} \cap A_{12}) \neq \mathbf{P}(A_{12})\mathbf{P}(A_{13})\mathbf{P}(A_{23})$, so the events are not jointly independent.

(c) The letters E,E,P,P,P,R are stamped on tiles and put in a bag. They are drawn from the bag without replacement one by one at random. What is the probability that the letters are drawn in the sequence P,E,P,P,E,R? How many different permutations of letters are there that begin with E?

Solution: Assume each tile has a number next to it, e.g. $E_1, E_2, P_1, P_2, P_3, R_1$. There are 6! total ways to order the numbered tiles, each equally likely. There are 3!2! ways to rearrange the P_i and E_i letters while still spelling $PEPPER$, so the desired event has cardinality 3!2!. Thus the probability is $3!2!/6! = 1/60$. This can also be done using the multiplication rule: there is probability 3/6 of getting a P first, then 2/5 of getting an E, etc.

For the second part, if the permuation begins with E, then we are counting unique arrangements of the remaining letters. There are 5! labelled permutations on the 5 remaining letters, but 3! rearrangements of the three P's actually give rise to the same word so the desired number is $5!/3! = 20$.

(d) Alice chooses a random number x uniformly between 0 and 1. Bob then picks independent random numbers $y_1, y_2 ...,$ uniformly from $[0, 1]$ until he has picked a number $y_L > x$. (So $y_i < x$ for $i = 1, 2, ..., L - 1$.) Bob is then given $L - 1$ dollars. What are Bob's expected winnings, I.e., what is $E[L - 1]$? HINT: What is $P(L > n)$?

Solution: Note that $L \geq n$ exactly when x is the largest among $y_1, \ldots, y_{n-1}$. But $x, y_1, \ldots, y_{n-1}$ are arranged according to a random permuation since they are i.i.d. continuous (perhaps surprisingly this holds for any continuous distribution as long as the rv's are i.i.d.), which implies that $\mathbf{P}(L \geq n) = \frac{1}{n}$. Now

$$\mathrm{E}\, L = \sum_{n \geq 1} \mathbf{P}(L \geq n) = \sum \frac{1}{n} = \infty\,.$$

But implies that $\mathrm{E}[L-1] = \mathrm{E}\, L - 1 = \infty$ as well.

You can also do this using the total expectation theorem. Given the value for the r.v. x, L is a geometric r.v. with parameter $1-x$. But x is itself random so we have to integrate against the desnity of x to get

$$\mathrm{E}\, L = \int_0^1 \mathrm{E}[L|x]dx = \int_0^1 \frac{1}{1-x}dx = \ln x\Big|_0^1 = \infty\,.$$

(e) Suppose there are n pairs of socks in the dryer and you pull out k socks at random. What is the average number of pairs will you have?

Solution: Let X_i be the indicator r.v. equal to 1 if we picked both socks from pair i and 0 otherwise. The number of sets of k socks containing pair i is given by $\binom{2n-2}{k-2}$, and the total number of sets of k socks is given by $\binom{2n}{k}$. Hence

$$\mathbf{P}(X_i = 1) = \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}} = \frac{k(k-1)}{(2n)(2n-1)}\,.$$

The number of pairs of socks is $X = \sum X_i$ and the X_i have the same distribution (but not independent!), so by linearity of expectation

$$\mathrm{E}\, X = n\, \mathrm{E}\, X_i = \frac{k(k-1)}{2(2n-1)}\,.$$

(f) If independent trials each resulting in success with probability p are performed, what is the probability of r successes before m failures?

Solution: Let's start by looking at an example. Denote by 1 a success and 0 a failure, and suppose $r = 2, m = 2$. Then possible sequences we want to consider are 011, 101, 11. The first two have probability $p^2(1-p)$ while the third has probability p^2. Another way to think of this is that we can have sequences of length at most $r+m$ (=4 here), and the good sequences are 011?, 101?, 11??, where ? means either 0 or 1 is OK. The point is that once you reach r success, you don't actually care what the remaining outcomes are.

Let the good set be A, the possible sequences with r successes before m failures. We partition A according to how many failures, l, $0 \leq l \leq m-1$ occur before the rth success. One such sequence, with r successes and l failures has probability $p^r(1-p)^l$ of occuring. How many such sequences are there? Notice that the final outcome, in position $r+l$, must be a success (convince yourself of this). Then there are l failures to put in the first $r+l-1$ positions, so there are $\binom{r+l-1}{l}$ of these sequences with l failures before the rth success. Putting it all together we get

$$\mathbf{P}[A] = \sum_{l=0}^{m-1} \mathbf{P}[A|l] = \sum_{l=0}^{m-1} \binom{r+l-1}{l} p^r(1-p)^l\,.$$

Problem 3 [15] Two jars contain 4 balls each. Jar 1 has red balls and Jar 2 has blue balls. A "switch" consists of randomly selecting a ball from each jar and simultaneously moving it into the other jar.

(a) For a given ball, b, let S_b be the total number of times it is selected in 4 switches. Find the pmf and expected value of S_b.

Solution: Note a switch involves choosing a ball from *each* jar. Thus the ball b has probability $1/4$ of being selected on each of the four switches, independently. Thus $S_b \sim \text{Binomial}(4, 1/4)$, and $\mathrm{E}\, S_b = 1$.

(b) What is the expected number of balls that are in their original jars after 4 switches?

Solution: The ball b is in its original jar if it participated in an even number of switches, i.e. $S_b = 0, 2,$ or 4. Let X_i be the indicator for ball i $(1 \le i \le 8)$ is in its original jar. Then

$$\mathbf{P}(X_i = 1) = (\frac{3}{4})^4 + \binom{4}{2}(\frac{1}{4})^2(\frac{3}{4})^2 + (\frac{1}{4})^4 = \frac{17}{32}\,.$$

Thus the expected number of balls that are in their original jars after 4 switches is given by

$$\mathrm{E}\sum_{i=1}^{8} X_i = 8\,\mathrm{E}\,X_i = \frac{17}{4}\,.$$

Problem 4 [20] Alice visits an island where there are two kinds of people: Alphas and Betas. Alpha's tell the truth with probability p_1 and Betas tell the truth with probability p_2. The probability that a randomly chosen person is an Alpha is α. Note that the responses from islanders are independent of each other.

(a) Alice cannot tell the two types apart and so decides to test a randomly chosen islander with questions she knows the answer to. She asks this person n questions and receives k correct responses. What is the probability that she has chosen an Alpha?

Solution: Let A be the event that she chose an Alpha. Then by Bayes' rule we have

$$\begin{aligned}\mathbf{P}(A|k \text{ correct out of } n) &= \frac{\mathbf{P}(k \text{ correct out of } n|A)\mathbf{P}(A)}{\mathbf{P}(k \text{ correct out of } n)} \\ &= \frac{\left(\binom{n}{k}p_1^k(1-p_1)^{n-k}\right)\alpha}{\left(\binom{n}{k}p_1^k(1-p_1)^{n-k}\right)\alpha + \left(\binom{n}{k}p_2^k(1-p_2)^{n-k}\right)\beta} \\ &= \frac{1}{1 + \frac{\beta}{\alpha}\left(\frac{p_2}{p_1}\right)^k\left(\frac{1-p_2}{1-p_1}\right)^{n-k}}\,.\end{aligned}$$

You can check that this probability behaves as you'd hope with respect to various limits, e.g. $\alpha \to 0$, or $\beta \to 0$, or setting $p_1 = p_2$, etc.

(b) Alice chooses another person at random and asks him a "Yes/No" question. The person answers "Yes". She then asks another randomly chosen person if the first person was telling the truth and this person says "Yes". What is the probability that the first person told the truth?

Solution: Let A be the event that the first person was telling the truth, and B the event that the second person says "yes, first person is truthful". Then we wish to find

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B)}\,.$$

Now a random person has probability $p = \alpha p_1 + (1-\alpha)p_2$ of telling the truth. Hence $\mathbf{P}(A) = p$, and $\mathbf{P}(B|A) = p$. Now B occurs exactly if person 1 is lying and person 2 lies as well, or person 1 is truthful and so is person 2, so $\mathbf{P}(B) = \mathbf{P}(\text{lie,lie}) + \mathbf{P}(\text{truth,truth}) = (1-p)^2 + p^2$. Putting it all together we get

$$\mathbf{P}(A|B) = \frac{p^2}{(1-p)^2 + p^2}\,.$$

As a basic check, note that if $p \to 0$ then this probability goes to zero, which makes sense – the two people are very likely to be both lying.

(c) Alice finally figures out how to tell the Alphas and Betas apart. She asks a fixed "Yes/No" question of N different Alphas. Given that they all give her the same answer, what is the probability that they are being truthful?

Solution: Let A be the event that they all give her the same answer, let T be the event that they are all being truthful, and let L be the event that they are all lying. We wish to find $\mathbf{P}(T|A)$. Note that $A = L \cup T$ since they all give the same answer if and only if they either all lie or all tell the truth. By definition of conditional probability and independence of the individuals' responses we have

$$\mathbf{P}(T|A) = \frac{\mathbf{P}(A \cap T)}{\mathbf{P}(A)} = \frac{\mathbf{P}(T)}{\mathbf{P}(A)} = \frac{\mathbf{P}(T)}{\mathbf{P}(T) + \mathbf{P}(L)} = \frac{p_1^N}{p_1^N + (1-p_1)^N}\,.$$