UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 2 Solutions
Spring 2011

Solution to Problem 1. Let e_1, e_2 denote the eye color of each parent, and let g_1, g_2 denote the genotypes of each parent. Then, by the Total Probability Theorem (TPT), the probability that two brown eyed parents have a blue eyed child is equal to

$$\begin{aligned}
&\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}) \\
&= \mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = BB, g_2 = BB) \cdot \mathbf{P}(g_1 = BB, g_2 = BB|e_1 = e_2 = \text{brown}) \\
&+ \mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = Bb, g_2 = BB) \cdot \mathbf{P}(g_1 = Bb, g_2 = BB|e_1 = e_2 = \text{brown}) \\
&+ \mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = BB, g_2 = Bb) \cdot \mathbf{P}(g_1 = BB, g_2 = Bb|e_1 = e_2 = \text{brown}) \\
&+ \mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = Bb, g_2 = Bb) \cdot \mathbf{P}(g_1 = Bb, g_2 = Bb|e_1 = e_2 = \text{brown}) \\
&+ \mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = bb \text{ or } g_2 = bb) \cdot \mathbf{P}(g_1 = bb \text{ or } g_2 = bb|e_1 = e_2 = \text{brown})
\end{aligned}$$

Now,

$$\mathbf{P}(g_1 = bb \text{ or } g_2 = bb|e_1 = e_2 = \text{brown}) = 0\,,$$

since neither parent can have genotype bb and brown eyes, so the last line evaluates to zero. Turning to the first four lines, we have

$$\begin{aligned}
\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = Bb, g_2 = BB) &= 0 \\
\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = BB, g_2 = Bb) &= 0 \\
\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = Bb, g_2 = Bb) &= 0 \\
\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}, g_1 = Bb, g_2 = Bb) &= \frac{1}{2} \cdot \frac{1}{2} = 1/4\,,
\end{aligned}$$

where the last probability is 1/4 because each parent must pass on a b gene, which occurs with probability 1/2 each, independently. Thus the desired probability is

$$\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}) = \frac{1}{4} \cdot \mathbf{P}(g_1 = Bb, g_2 = Bb|e_1 = e_2 = \text{brown})\,.$$

We now use Bayes' rule to get

$$\begin{aligned}
\mathbf{P}(g_1 = Bb, g_2 = Bb|e_1 = e_2 = \text{brown}) &= \mathbf{P}(e_1, e_2 = \text{brown}|g_1 = Bb, g_2 = Bb) \cdot \frac{\mathbf{P}(g_1 = Bb, g_2 = Bb)}{\mathbf{P}(e_1, e_2 = \text{brown})}\,. \\
&= 1 \cdot \frac{\mathbf{P}(g_1 = Bb)\mathbf{P}(g_2 = Bb)}{\mathbf{P}(e_1 = \text{brown})\mathbf{P}(e_2 = \text{brown})}\,,
\end{aligned}$$

where the last step follows by independence of the two parents' genotypes and phenotypes (you might debate this, arguing that perhaps people are attracted to others' of the same eye color, but we ignore this effect here). We are given that $\mathbf{P}(g_1 = Bb) = \gamma$ (and similarly for g_2), and by the TPT,

$$\begin{aligned}\mathbf{P}(e_1 = \text{brown}) = \mathbf{P}(e_1 = \text{brown}|g_1 = BB)\mathbf{P}(g_1 = BB) \\ + \mathbf{P}(e_1 = \text{brown}|g_1 = Bb)\mathbf{P}(g_1 = Bb) \\ + \mathbf{P}(e_1 = \text{brown}|g_1 = bb)\mathbf{P}(g_1 = bb) \\ = 1 \cdot \alpha + 1 \cdot \gamma + 0 \cdot \beta = \alpha + \gamma\,.\end{aligned}$$

Thus, plugging in, the final result is

$$\mathbf{P}(\text{BlueChild}|e_1 = e_2 = \text{ brown}) = \frac{1}{4}\frac{\gamma^2}{(\alpha+\gamma)^2}\,.$$

We can check the answer by seeing what happens if $\gamma = 1$ and $\alpha = 0$: in this case everyone in the population has genotype Bb, and indeed the probability of having a blue eyed child is $1/4$. If, however, $\alpha = 1$ and $\gamma = 0$, then everyone in the population has genotype BB and the probability of getting a blue eyed child is zero.

For the second part we are to find the probability that a blue eyed and brown eyed couple have a blue eyed child, i.e. $\mathbf{P}(\text{BlueChild}|e_1 = \text{blue}, e_2 = \text{ brown})$. We know that $\{e_1 = \text{blue}\} = \{g_1 = bb\}$, so as before by the TPT we have

$$\begin{aligned}&\mathbf{P}(\text{BlueChild}|e_1 = \text{blue}, e_2 = \text{ brown}) \\ &= \mathbf{P}(\text{BlueChild}|g_1 = bb, e_2 = \text{brown}) \\ &= \mathbf{P}(\text{BlueChild}|g_1 = bb, e_2 = \text{brown}, g_2 = BB)\mathbf{P}(g_2 = BB|e_2 = \text{brown}) \\ &\quad + \mathbf{P}(\text{BlueChild}|g_1 = bb, e_2 = \text{brown}, g_2 = Bb)\mathbf{P}(g_2 = Bb|e_2 = \text{brown})\,.\end{aligned}$$

The first probability in the first term is zero, since the second parent passes along a B gene and the child then has brown eyes. Now,

$$\mathbf{P}(\text{BlueChild}|g_1 = bb, e_2 = \text{brown}, g_2 = Bb) = \frac{1}{2}\,,$$

since the second parent has probability $1/2$ of passing on the b gene. Finally, as before we have by Bayes' rule that

$$\mathbf{P}(g_2 = Bb|e_2 = \text{brown}) = \frac{\mathbf{P}(g_2 = Bb)}{\mathbf{P}(e_2 = \text{brown})} = \frac{\gamma}{\alpha+\gamma}\,.$$

This gives the desired probability

$$\mathbf{P}(\text{BlueChild}|e_1 = \text{blue}, e_2 = \text{ brown}) = \frac{1}{2} \cdot \frac{\gamma}{\alpha+\gamma}\,.$$

Solution to Problem 2.

(a) We check the three axioms of probability:
1) For any $A \subseteq \Omega$,

$$Q(A) = \mathbf{P}(A|B) = \mathbf{P}(A \cap B)/\mathbf{P}(B) \geq 0$$

since $\mathbf{P}$ is itself a probability law so $\mathbf{P}(S) \geq 0$ for any S (in particular for $S = A \cap B$).
2) We have that

$$Q(\Omega) = \mathbf{P}(\Omega \cap b)/\mathbf{P}(B) = \mathbf{P}(B)/\mathbf{P}(B) = 1 \,.$$

3) If $A_1, A_2 \subseteq \Omega$ are mutually exclusive, then we have

$$\begin{aligned} Q(A_1 \cup A_2) &= \frac{\mathbf{P}(A_1 \cup A_2 \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}((A_1 \cap B) \cup (A_2 \cap B))}{\mathbf{P}(B)} \\ &= \frac{\mathbf{P}(A_1 \cap B) + \mathbf{P}(A_2 \cap B)}{\mathbf{P}(B)} = \mathbf{P}(A_1|B) + \mathbf{P}(A_2|B) \\ &= Q(A_1) + Q(A_2) \,, \end{aligned}$$

where we used the fact that $(A_1 \cap B)$ and $(A_2 \cap B)$ are mutually exclusive.

(b) Note that we may use the definition of conditional probability for Q since Q is a probability law in its own right. Thus,

$$Q(A|C) = \frac{Q(A \cap C)}{Q(C)} = \frac{\mathbf{P}(A \cap C|B)}{\mathbf{P}(C|B)} = \frac{\mathbf{P}(A \cap C \cap B)\mathbf{P}(B)}{\mathbf{P}(B)\mathbf{P}(C \cap B)} = \mathbf{P}(A|B \cap C) \,.$$

With this notation, we are supposed to show that

$$Q(A|C) = \frac{Q(C|A)Q(A)}{Q(C)} \,,$$

but this is just Bayes' rule for Q! So we're done.

Solution to Problem 3.

(a) In order to wind up in the same place after two steps, the tightrope walker can either step forwards, then backwards, or vice versa. Therefore the required probability is:

$$2 \cdot p \cdot (1 - p).$$

(b) The probability that after three steps he will be one step ahead of his starting point is the probability that out of 3 steps in total, 2 of them are forwards, and one is backwards. This equals:

$$\binom{3}{1} \cdot p^2 \cdot (1 - p).$$

(c) Given that out of his three steps only one is backwards, the sample space for the experiment is:

$$\{(F, F, B); (F, B, F); (B, F, F)\}$$

where F denotes a step forwards, and B a step backwards. Each of these sample points is equally likely, therefore the probability that his first step is a step forward is $\frac{2}{3}$.

Solution to Problem 4.

(a) We need to consider the sample space. The sample space is the space of ordered pairs (x_1, x_2) with $1 \leq x_1, x_2 \leq 6$. Each point in the sample space is equally likely, and there are 6 ”favorable” outcomes, hence the probabability of doubles is $\frac{1}{6}$.

(b) Once we are told that the sum is under 4, our sample space changes to:

$$\{(1,1); (1,2); (1,3); (2,1); (2,2); (3,1)\}$$

and hence the probability of doubles is $\frac{1}{3}$.

Solution to Problem 5. Initially, 10 forks and no knives in the left drawer, which we denote by (10F,0K). Similarly, right drawer has (0F,10K). After the roommate takes two forks out of the left drawer and places them in the right drawer, the composition of the two drawers becomes (8F,0K) and (2F,10K). From the right drawer, the probability of pulling a fork is $2/12 = 1/6$, and the probability of pulling a knife is $5/6$.
From the initial configuration then, the system may have evolved in two possible ways. We call these cases C_1 and C_2. Case C_1 refers to a fork being transferred from the right drawer to the left, leaving the two drawers as (9F,0K) and (1F,10K), C_2 refers to a knife transferred to the left, resulting in (8F,1K) and (2F,9K).
From the above argument, $\mathbf{P}(C_1) = 1/6$, and $\mathbf{P}(C_2) = 5/6$.
From Bayes's Rule,

$$\mathbf{P}(\text{left chosen}|\text{knife pulled}) = \frac{\mathbf{P}(\text{knife pulled}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife pulled})}$$

Using the Total Probability Theorem,

$$\begin{aligned}\mathbf{P}(\text{knife}|\text{left chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{left chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{left chosen})\mathbf{P}(C_2) \\ &= 0 \cdot \frac{1}{6} + \frac{1}{9} \cdot \frac{5}{6} = \frac{5}{54}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbf{P}(\text{knife}|\text{right chosen}) &= \mathbf{P}(\text{knife}|C_1, \text{right chosen})\mathbf{P}(C_1) + \mathbf{P}(\text{knife}|C_2, \text{right chosen})\mathbf{P}(C_2) \\ &= \frac{10}{11} \cdot \frac{1}{6} + \frac{9}{11} \cdot \frac{5}{6} = \frac{5}{6}\end{aligned}$$

Using Bayes's Theorem,

$$\begin{aligned}\mathbf{P}(\text{left chosen}|\text{knife}) &= \frac{\mathbf{P}(\text{knife}|\text{left chosen})\mathbf{P}(\text{left chosen})}{\mathbf{P}(\text{knife})} \\ &= \frac{5/54 \cdot 1/2}{5/54 \cdot 1/2 + 5/6 \cdot 1/2} \\ &= \frac{1}{10}\end{aligned}$$

Solution to Problem 6. Without prior bias on whether the exit of campus lies East or West, the exact answers of the passerby are not as important as whether a string of answers is similar or not. Let R_r denote the event that we receive r similar answers and T denote the event that these repeated answers are truthful. Let S denote the event that the questioned passerby is a student. Note that, because a professor always gives a false answer, $T \cap S^c = \emptyset$ and thus $\mathbf{P}(T \cap S^c) = 0$. Therefore,

$$\mathbf{P}(T|R_r) = \frac{\mathbf{P}(T \cap R_r)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r \cap S)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r|S)\mathbf{P}(S)}{\mathbf{P}(R_r)}$$

where the stated independence of a passerby's successive answers implies $\mathbf{P}(T \cap R_r|S) = \left(\frac{3}{4}\right)^r$. Applying the Total Probability Theorem and again making use of independence, we also deduce

$$\mathbf{P}(R_r) = \mathbf{P}(R_r|S)\mathbf{P}(S) + \underbrace{\mathbf{P}(R_r|S^c)}_{1}\mathbf{P}(S^c) = \left(\left(\frac{3}{4}\right)^r + \left(\frac{1}{4}\right)^r\right)\frac{2}{3} + \frac{1}{3} \quad .$$

(a) Applying the above formulas for $r = 1$, we have $\mathbf{P}(R_1) = 1$ and thus

$$\mathbf{P}(T|R_1) = \frac{\frac{3}{4} \cdot \frac{2}{3}}{1} = \boxed{\frac{1}{2}} \quad .$$

(b) For $r = 2$, the formulas yield $\mathbf{P}(R_2) = \frac{3}{4}$ and thus

$$\mathbf{P}(T|R_2) = \frac{\left(\frac{3}{4}\right)^2 \frac{2}{3}}{\frac{3}{4}} = \boxed{\frac{1}{2}}.$$

(c) For $r = 3$, the formulas yield $\mathbf{P}(R_3) = \frac{15}{24}$ and thus

$$\mathbf{P}(T|R_3) = \frac{\left(\frac{3}{4}\right)^3 \frac{2}{3}}{\frac{15}{24}} = \boxed{\frac{9}{20}} \quad .$$

(d) For $r = 4$, the formulas yield $\mathbf{P}(R_4) = \frac{35}{64}$ and thus

$$\mathbf{P}(T|R_4) = \frac{\left(\frac{3}{4}\right)^4 \frac{2}{3}}{\frac{35}{64}} = \boxed{\frac{27}{70}} \quad .$$

(e) As soon as we receive a dissimilar answer from the same passerby, we know that this passerby is a student; a professor will always give the same (false) answer. Let D denote the event of receiving the first dissimilar answer. Given D on the fourth answer, either the student has provided three truthful answers followed by one untruthful answer, occurring with probability $\left(\frac{3}{4}\right)^3 \frac{1}{4}$, or the student has provided three untruthful answers followed by one truthful answer, occurring with probability $\left(\frac{1}{4}\right)^3 \frac{3}{4}$. Note that event T corresponds to the former; thus,

$$\mathbf{P}(T|R_3 \cap D) = \frac{\left(\frac{3}{4}\right)^3 \frac{1}{4}}{\left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{1}{4}\right)^3 \frac{3}{4}} = \boxed{\frac{9}{10}} \quad .$$

In parts (a) - (d), notice the decreasing trend in the probability of the passer-by being truthful as the number of similar answers grows. Intuitively, our confidence that the passerby is a professor grows as the sequence of similar answers gets longer, because we know a professor will always give the same (false) answer while a student has a chance to answer either way. However, as part (e) demonstrates, the first indication that the passerby is a student will boost our confidence that the previous string of similar answers are truthful, because any single answer by the student has a 3-to-1 chance of being a truthful one.
For the remainder of this problem, let E and W represent the events that a passerby provides East and West, respectively, as an answer and let T_E represent the event that East is the correct answer. We are told Ima's a-priori bias is $\mathbf{P}(T_E) = \epsilon$.

(a) Using Bayes's Rule and all the arguments used in parts (a) - (e), we have

$$\begin{aligned}\mathbf{P}(T_E|E) &= \frac{\mathbf{P}(E|T_E)\mathbf{P}(T_E)}{\mathbf{P}(E)} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and} \\ \mathbf{P}(T_E|W) &= \frac{\mathbf{P}(W|T_E)\mathbf{P}(T_E)}{\mathbf{P}(W)} = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)(1-\epsilon)} = \boxed{\epsilon} \quad .\end{aligned}$$

In particular, we have used that $\mathbf{P}(E) = \mathbf{P}(E|T_E)\mathbf{P}(T_E) + \mathbf{P}(E|T_E^c)\mathbf{P}(T_E^c)$ (and similarly for $\mathbf{P}(W)$)

(b) Likewise, given two consecutive and similar answers from the same passerby, we have

$$\begin{aligned}\mathbf{P}(T_E|EE) &= \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and} \\ \mathbf{P}(T_E|WW) &= \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2(1-\epsilon)} = \boxed{\epsilon} \quad .\end{aligned}$$

(c) Finally, given three consecutive and similar answers from the same passerby,

$$\begin{aligned}\mathbf{P}(T_E|EEE) &= \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\frac{9\epsilon}{11-2\epsilon}} \quad \text{and} \\ \mathbf{P}(T_E|WWW) &= \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3(1-\epsilon)} = \boxed{\frac{11\epsilon}{9+2\epsilon}} \quad .\end{aligned}$$

For $\epsilon = \frac{9}{20}$, we calculate $\mathbf{P}(T_E|EEE) = \frac{81}{202}$ and $\mathbf{P}(T_E|WWW) = \frac{1}{2}$.

Notice that the E, EE and EEE answers to parts (f) - (h) match the answers to parts (a)-(c) when $\epsilon = \frac{1}{2}$, or when Ima's prior bias does not favor either possibility.