UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 1 Solutions
Spring 2011

Solution to Problem 1.

(a) $A \cap B \cap C$

(b) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

(c) $(A^c \cap B^c \cap C^c) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

(d) $[A^c \cap B^c \cap C^c]^c$

(e) $A^c \cap B^c \cap C^c$

(f) $A^c \cap B \cap C$

(g) $C \cup B^c$

The Venn diagrams are not included here.

Solution to Problem 2. We have that

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)\,.$$

Now, $\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1 - 0.7 = 0.3$. Thus

$$\mathbf{P}(A \cup B) = 0.3 + 0.2 - 0.1 = 0.4\,.$$

Solution to Problem 3. We could have a two-dimensional sample space containing 52^2 points, where each axis represents a particular card. However, this sample space would be finer grain than necessary to determine the desired probabilities.
For parts (a) and (c), we can ignore the suit of the cards. Letting the cards be denoted by $1, 2, \ldots, 13$, we have the sample space $\Omega = \{(i, j) : 1 \le i, j \le 13\}$, where $|\Omega| = 169$ points representing the 169 possible outcomes. The probability law is uniform on Ω.
Define event A to be when Masha draws an ace, event S to be when Sasha draws an ace. Then we know that

$$\mathbf{P}(A) = \mathbf{P}(B) = \frac{1}{13}$$

and

$$\mathbf{P}(A \cap B) = \frac{1}{169}\,.$$

For part (a) we have

$$\mathbf{P}(\text{at least one card is an ace}) = \mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \frac{25}{169}$$

and for part (c) we have

$$\mathbf{P}(\text{neither card is an ace}) = 1 - \mathbf{P}(\text{at least one card is an ace}) = \frac{144}{169}\,.$$

For parts (b) and (d), since we are only interested in the suits of the cards, we represent the sample space as the following 16 points. The horizontal axis represents the suit of Masha's card, and the vertical axis represents the suit of Sasha's card. Each of the points is equally likely; therefore, the probability of any particular point occurring is $\frac{1}{16}$. Let A be the event that the cards have the same suit, and B be the event that neither card is a diamond or club. Then

$$\mathbf{P}(A) = |A|/|\Omega| = 4/16 = 1/4\,,$$

where we used that $|A| = 4$ since A consists exactly of the diagonal points $(1,1),(2,2),(3,3),(4,4)$. For (d), let the suit of diamond be denoted by the number 4 and the suit of clubs by the number 3. The event B is therefore obtained from Ω by removing all points with any coordinate 3 or 4, leaving only the points $(1,1),(1,2),(2,1),(2,2)$. We get that $\mathbf{P}(B) = |B|/|\Omega| = 4/16 = 1/4$.

Solution to Problem 4. Without knowing any other information, it makes sense for the sample space to include every possible outcome of the experiment, i.e. the actual possible sequences of rolls. Since the process terminates if a 2 or a 4 is rolled, the sample space consists of arbitrary finite sequences of 1 and 3, terminated by either a 2 or a 4, and also infinite sequences of 1s and 3s. More precisely, write $\Omega_2 = \{1,3\}^*|2$ for the set of arbitrary finite sequences of 1s and 3s terminated by a 2 and similarly for Ω_4. Also, let $\Omega_\infty = \{1,3\}^\infty$ denote infinite length sequences of 1s and 3s. Then

$$\Omega = \Omega_2 \cup \Omega_4 \cup \Omega_\infty\,.$$

Solution to Problem 5.

(a) We know that

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

Rearranging, we get

$$\mathbf{P}(A \cap B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cup B).$$

Since $(A \cup B)$ is always a subset of Ω, the universal event, therefore, $\mathbf{P}(A \cup B) \leq \mathbf{P}(\Omega)$ and

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(\Omega).$$

Finally, by the normalization axiom, $\mathbf{P}(\Omega) = 1$ and

$$\mathbf{P}(A \cap B) \geq \mathbf{P}(A) + \mathbf{P}(B) - 1.$$

(b) We begin by writing

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P((A^c \cap B) \cup (A \cap B^c)) \\ &= P(A^c \cap B) + P(A \cap B^c), \end{aligned}$$

where the last equality is from the additivity axiom. Next, we know that $B = (A^c \cap B) \cup (A \cap B)$ and $(A^c \cap B) \cap (A \cap B) = \emptyset$ so that we may apply the additivity axiom to get

$$P(B) = P(A^c \cap B) + P(A \cap B).$$

With rearrangement, this becomes

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

By symmetry, we also have

$$P(B^c \cap A) = P(A) - P(A \cap B).$$

So plugging in for $P(A^c \cap B)$ and $P(B^c \cap A)$, we get

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P(B) - P(A \cap B) + P(A) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

Solution to Problem 6. The easiest way to solve this problem is to make a table of some sort, similar to the one below.

Die 1	Die 2	Sum	P(Sum)
1	1	2	2p
1	2	3	3p
1	3	4	4p
1	4	5	5p
2	1	3	3p
2	2	4	4p
2	3	5	5p
2	4	6	6p
3	1	4	4p
3	2	5	5p
3	3	6	6p
3	4	7	7p
4	1	5	5p
4	2	6	6p
4	3	7	7p
4	4	8	8p
		Total	80p

$$\begin{aligned} \mathbf{P}(\text{All events}) &= 1 \\ &= 80p \quad \text{(Total from the table)} \\ \Rightarrow p &= \frac{1}{80} \end{aligned}$$

(a)

$$\begin{aligned}\mathbf{P}(\text{sum being even}) &= 2p+4p+4p+6p+4p+6p+6p+8p\\ &= 40p\\ &= \boxed{1/2}\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{P}(\text{rolling a 4 and a 1}) &= \mathbf{P}(1,4)+\mathbf{P}(4,1)\\ &= 5p+5p\\ &= 10p\\ &= \boxed{1/8}\end{aligned}$$

Solution to Problem 7. This problem was a bit tricky, so don't be discouraged if you didn't completely get it. Just think through the solution carefully.

(a) Let us denote a heads by a 1 and a tails by a 0. Then Ω_∞ is just the set of infinite binary sequences. The basic idea is to think of numbers in $[0,1]$ in their binary representation, for example $0.25 = 0.01$ in binary. The problem is a bit subtle though, since for example the two binary sequences 0.100... and .011... both represent 1/2. We next describe one way to overcome this technical issue, but the issue is ignored in the remaining parts of the problem.

To deal with this issue, let $\mathbb{Q}_2 \subset [0,1]$ denote the set of numbers which can be represented with finite binary sequences (these are often called the "Dyadic Rationals"). Consider the set $A = \frac{1}{2}\cdot\mathbb{Q}_2 \subset \mathbb{Q}_2$, which is the set of dyadic rationals shifted over by one binary point (the dyadic rational 0.101 gets included in A as 0.0101), and the set $B = \frac{1}{2}+\frac{1}{2}\cdot\mathbb{Q}_2 \subset \mathbb{Q}_2$, which is the set of dyadic rationals shifted over by one binary point with a 1 inserted right after the point (0.101 gets included in B as 0.1101). Then we can make a bijection between the set A and all binary sequences having only a finite number of 1s by ignoring the first zero in the binary representation of numbers in A. Similarly, we can make a bijection between the set B and all binary sequences terminated with an infinite number of 1s by ignoring the first 1 in the binary representation of numbers in B. All numbers in $[0,1]$ except for the dyadic rationals have a unique binary representation (prove this). Thus we have a bijection between $A \cup B \cup ([0,1] \setminus \mathbb{Q}_2)$ and infinite binary sequences. (As a similar but simpler concept, think about the fact that the set of integers $\mathbb{Z}$ is in one-one correspondence with the even integers $2\mathbb{Z}$ and also in one-one correspondence with the odd integers $2\mathbb{Z}+1$. But $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z}+1)$, so the integers contain two copies of themselves! We do the same thing to make a correspondence between the subset $\mathbb{Q}_2 \subset [0,1]$ and the *two* copies of $\mathbb{Q}_2$ in Ω_∞.)

(b) For the remainder of the problem we will ignore the technical issue discussed in part (a). If the first five flips are heads and the rest are unknown, this corresponds to a binary representation 0.11111???.... Where the ? are either a 0 or a 1. We can also write this as $.11111|\{0,1\}^\infty$, i.e. the sequence 0.11111 followed by an infinite string of 0's and 1's. The binary number 0.11111 is equal to $1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 31/32$. This means that five flips corresponds to the interval $[31/32, 1)$.

(c) For an infinite sequence x in Ω_∞ we write $b(x) \in [0,1]$ for the corresponding binary representation. The natural probability assigned to a subset $B \subseteq [0,1]$ is the length of the subset, i.e. $\mathbf{P}_{[0,1]}(B) = \int_B dx$. Thus $\mathbf{P}(\text{five heads}) = \mathbf{P}(A) = \mathbf{P}_{[0,1]}(b(A)) = \mathbf{P}_{[0,1]}([31/32, 1)) = 1/32$.

(d) The mappings in parts (d) and (e) are not functions, but set-valued functions, with a singe element being mapped to a set. Let finite sequenes of coin flips be encoded by binary strings, zero for tails and one for heads. For a length n string, we map to the interval

$$\left[\sum_{i=1}^{n-1} \frac{1}{2^i}, \sum_{i=1}^{n} \frac{1}{2^i}\right) = [2(1-2^{-n}) - 1, 2(1-2^{-n-1}) - 1) = [1 - 2^{-n+1}, 1 - 2^{-n}) \,.$$

For example, all length two sequences will get mapped to the interval $[1/2, 3/4)$. We now describe this mapping. For $x = x_1, x_2, x_3, \ldots, x_n \in \Omega_{\text{finite}}$ (here $x_i \in \{0,1\}$), let $g : \Omega_{\text{finite}} \to [0,1]$ be the map

$$x \mapsto \left[(1 - 2^{-n+1}) + 2^{-n} \sum_{i=1}^{n} \frac{x_i}{2^i}, (1 - 2^{-n+1}) + 2^{-n} \left(\sum_{i=1}^{n} \frac{x_i}{2^i} + \frac{1}{2^n}\right)\right) \,.$$

Note that this mapping assigns the natural probability law (obtained from the one on $[0,1]$ using our mapping g) to finite sequences of coin flips: probability 2^{-2n} for each length-n sequence; this means probability 2^{-n} of getting a length n sequence, and conditioned on getting a length n sequence the probability is uniform among these sequences.

(e) We define the mapping $h : \Omega_{\text{finite}} \to \Omega_\infty$ by $h = b^{-1} \circ g$. This means first map to the interval using the map from (d), then take the (infinite) binary expansion. But this map is actually quite simple! For a given finite binary string $x \in \Omega_{\text{finite}}$, we put $n-1$ 1s in front of the string, and then put x, and finally concatenate all infinite binary strings at the end. For example, the string 11010 gets mapped to $0.11111|11010|\{0,1\}^\infty$, where the bars are added for clarity.