UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 9
Spring 2011

Issued: Wednesday, March 30, 2011 **Due:** Wednesday, April 6, 2011

Reading: Bertsekas & Tsitsiklis, Chapter 6

Problem 1. Each of n packages ($n \geq 1$) is loaded independently onto either a red truck, with probability p, or onto a green truck, with probability $1-p$. Let R be the total number of items selected for the red truck and let G be the total number of items selected for the green truck.

(a) Determine the PMF, expected value, and variance of the random variable R.

(b) Evaluate the probability that the first item to be loaded ends up being the only one on its truck.

(c) Evaluate the probability that at least one truck ends up with a total of exactly one package. (*Hint:* consider cases $n = 1$ and $n = 2$, separately)

(d) Evaluate the expectated value and the variance of the difference $R - G$.

(e) Assume that $n \geq 2$. Given that both of the first two packages to be loaded go onto the red truck, find the conditional expectation, variance, and PMF of the random variable R.

Problem 2. A train bridge is constructed across a wide river. Trains arrive at the bridge according to a Poisson process of rate $\lambda = 3$ per day.

(a) If a train arrives on day 0, find the probability that there will be no trains on days 1, 2, and 3.

(b) Find the probability that the first train to arrive, after the train on day 0, takes more than 3 days to arrive.

(c) Find the probability that no trains arrive in the first 2 days, but 4 trains arrive on the 4^{th} day.

(d) Find the probability that it takes more than 2 days for the 5^{th} train to arrive at the bridge.

Problem 3. On each trial of a game, both Don and Greg simultaneously flip biased coins. On each trial, the probability that Don's flip results in a head is p_d and the probability that Greg's flip results in a head is p_g, independent of the outcome of Don's flip. (Note that p_d and p_g are some fixed parameters throughout this problem. Your answers will be expressions in terms of p_d and p_g.)

(a) Given that the flips on a particular trial resulted in 2 heads, find the PMF for M, the number of additional trials up to and including the next trial on which 2 heads result.

(b) Given that the flips on a particular trial resulted in *at least* one head, find the probability that Don flipped a head on the trial.

(c) Starting from a trial on which no heads result, find the probability that Don's next flip of a head will occur *before* Greg's next flip of a head.

(d) Given that the event of the previous part has occurred (that is, Don's first flip of head occurred before that of Greg's) find the expected number of trials up to and including Don's first flip of head. Repeat for the expected number of trials to Greg's first flip of head (given the same event as before.)

Problem 4. Fred is giving out samples of dog food. He makes calls door to door, but he leaves a sample (one can) only on those calls for which the door is answered *and* a dog is in residence. On any call the probability of the door being answered is 3/4, and the probability that any household has a dog is 2/3. Assume that the events "Door answered" and "A dog lives here" are independent and also that the outcomes of all calls are independent.

(a) Determine the probability that Fred gives away his first sample on his third call.

(b) Given that he has given away exactly four samples on his first eight calls, determine the conditional probability that Fred will give away his fifth sample on his eleventh call.

(c) Determine the probability that he gives away his second sample on his fifth call.

(d) Given that he did not give away his second sample on his second call, determine the conditional probability that he will leave his second sample on his fifth call.

(e) We will say that Fred "needs a new supply" immediately *after* the call on which he gives away his last can. If he starts out with two cans, determine the probability that he completes at least five calls before he needs a new supply.

(f) If he starts out with exactly m cans, determine the expected value and variance of D_m, the number of homes with dogs which he passes up (because of no answer) before he needs a new supply.

Problem 5. Let $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ be two independent Poisson processes with rate parameters λ_1 and λ_2 respectively. These processes measure the number of customers arriving in stores 1 and 2 respectively.

(a) What is the probability that a customer arrives in store 1 before any customers arrive in store 2?

(b) What is the probability that in the first hour a total of exactly four customers have arrived at the two stores?

(c) Given that exactly four customers have arrived at the two stores, what is the probability that all four went to store 1?

(d) Let T denote the time of arrival of the first customer at store 2. Then X_T is the number of customers in store 1 at the time of the first customer arrival at store 2. Find the probability distribution of X_T (i.e. for each k find $\mathbb{P}(X_T = k)$).

Problem 6. Let $(N(t), t \geq 0)$ be a Poisson process of rate λ. That is, $N(t)$ is the number of Poisson arrivals in the interval $(0, t]$. Let T_1 be the first arrival time. Compute $\mathbb{E}(T_1 \mid N(1) = n)$. [*Hint:* $\mathbb{E}(Z) = \int_0^\infty P(Z > t)dt$ for any nonnegative random variable Z. Consider $n = 0$ and $n \geq 1$ separately.]

Problem 7. Consider a Poisson process $(N(t), t \geq 0)$ with rate λ. Let $\{B_1, \ldots, B_k\}$ be a partition of $[0, t]$. Let $\nu(B_i)$ be the number of Poisson arrivals in B_i for $i = 1, \ldots, k$. Let T_i be the i-th Poisson arrival.

(a) Find the conditional joint PMF of $(\nu(B_1), \ldots, \nu(B_k))$ given $N(t) = n$.

(b) Let $U_1, \ldots, U_n$ be i.i.d. from the uniform distribution on $[0, t]$. Let $\mu(B_i)$ be the number of $\{U_1, \ldots, U_n\}$ the lie in B_i. Compute the joint PMF of $(\mu(B_1), \ldots, \mu(B_k))$.

(c) By comparing results of parts (a) and (b), conclude that given $N(t) = n$, the unordered collection of arrival times $\{T_1, \ldots, T_n\}$ has the same distribution as that of $\{U_1, \ldots, U_n\}$.

(d) As an application of part (c), compute $\mathbb{E}\big(e^{s\sum_{i=1}^n T_i} \mid N(t) = n\big)$ for some fixed $s \in \mathbb{R}$. [Bonus: compute the moment generating function of $\sum_{i=1}^{N(t)} T_i$.]