UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 8

Spring 2011

Issued: Wed, March 16, 2011 **Due:** In HW box Wednesday, Mar 30, 2011

Reading: Bertsekas & Tsitsiklis, §5.1–5.4

Problem 1. You enter a castle with 78 floors. Your goal is to get to the treasure at the top. On each floor of the castle, there is a probability 1/2 of bumping into someone guarding that floor. Bumping into someone results in a change in your energy (the person you bump into may be friendly and give you something to eat, or may be nasty and you may have to put up a fight, etc). Your change in energy in each such interaction is distributed as a normal random variable with mean 1 unit and standard deviation 1/2 units, and is independent of how many people you bump into and of your change in energy in other interactions. No matter what your energy change on any given floor, you always proceed to the next floor. Let X be your total change in energy by the time you reach the treasure.

(a) Use the total probability law to find the PDF and transform associated with X. Is X normal? (HINT: First compute the transform of X conditioned on the number of guards you bumped into.)

(b) Find the transform associated with X by viewing X as a sum of a random number of random variables.

Problem 2. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $\mathrm{E}[X|Y]$ are positively correlated.

(b) Show that the correlation coefficient of Y and $\mathrm{E}[X|Y]$ has the same sign as the correlation coefficient of X and Y.

Problem 3. In general, linear estimators based on multiple measurements are not difficult to derive, but the algebra (and calculus) can become quite tedious. For this problem, derive the linear least squares estimator of X when two measurements are available. More specifically, find a_1, a_2, and b such that $g(Y_1, Y_2) = a_1Y_1 + a_2Y_2 + b$ minimizes

$$\mathrm{E}[(X - g(Y_1, Y_2))^2] = \mathrm{E}[(X - a_1Y_1 - a_2Y_2 - b)^2]$$

To make the problem a little more tractable, assume that Y_1 and Y_2 are uncorrelated (i.e., $\mathrm{E}[Y_1Y_2] = \mathrm{E}[Y_1]\,\mathrm{E}[Y_2]$).

Problem 4. Let S_n be the number of successes in n independent Bernoulli trials, where the probability of success in each trial is $p = 1/2$. Provide a numerical value for the limit as n tends to infinity for each of the following three expression

(a) $\mathbf{P}(\frac{n}{2} - 10 \leq S_n \leq \frac{n}{2} + 10)$

(b) $\mathbf{P}(\frac{n}{2} - \frac{n}{10} \leq S_n \leq \frac{n}{2} + \frac{n}{10})$

(c) $\mathbf{P}(\frac{n}{2} - \frac{\sqrt{n}}{2} \leq S_n \leq \frac{n}{2} + \frac{\sqrt{n}}{2})$

Problem 5. In this problem we investigate whether the Chebyshev inequality is tight. In particular, for every μ, σ, and $c \geq \sigma$, does there exist a random variable X with mean μ and standard deviation σ such that

$$\mathbf{P}(|X - \mu| \geq c) = \frac{\sigma^2}{c^2}\,?$$

Problem 6. We are laying out 25 plastic planks lengthwise, trying to make a path of about 1000 meters. The plastic planks are made in molds, and any variation in the lengths of the planks is due entirely to variation between different molds. The length in meters, X, of any particular mold used for making planks is independent of the length of all other molds. X is uniformly distributed between $40 - \sqrt{3}$ and $40 + \sqrt{3}$ meters. X has an expected value of 40 meters and a standard deviation of 1 meter. What is the probability that the resulting path will be within 1000 ± 7.5 meters if we use 25 planks ...

(a) ... all made from the same mold?

(b) ... each made from a different mold?

Explain the difference between the answers.

Problem 7. Let $X_1, X_2, \ldots$ be a sequence of independent random variables that are uniformly distributed between 0 and 1. For every n, we let Y_n be the median of the values $X_1, X_2, \ldots, X_{2n+1}$. (I.e., we order $X_1, \ldots, X_{2n+1}$ in increasing order and let Y_n be the $(n+1)$st element in this ordered sequence.) Show that the sequence Y_n converges to $1/2$ in probability.

Problem 8. Let $X_1, X_2, \ldots$ be a sequence of IID random variables with zero mean and finite variance $\text{var}(X_1) < \infty$, and denote by $\bar{X}_n$ the sample mean $\frac{X_1 + \cdots + X_n}{n}$. We learned in class that $\bar{X}_n \to 0$ in probability, or in other words for any $\epsilon > 0$ it holds that

$$\lim_{n \to \infty} \mathbf{P}(|\bar{X}_n| > 0) = 0, .$$

This is called the *weak law of large numbers*. The proof was very simple: compute $\text{var}(\bar{X}_n)$, and simply show that the variance goes to zero. The desired result then follows by Chebyshev's inequality.

In this problem we will investigate what happens if $\text{var}(X_i) = \infty$. We will prove the weak law of large numbers without the finite variance assumption, but still assuming $\mathrm{E}\,|X| < \infty$.

(a) Produce an example of a random variable X with with $\text{var}(X) = \infty$ but $\mathrm{E}\,|X| < \infty$. What is the variance of $\bar{X}_n$? Think about what the weak law says about $\bar{X}_n$, and whether it makes sense.

(b) One way to prove the weak law of large numbers when $\text{var}(X) = \infty$ is to consider *truncated* variables. Fix some positive integer M and define

$$Y_i^M = X_i 1_{|X_i|<M} = \begin{cases} X_i & \text{if } |X_i| \leq M \\ 0 & \text{if } |X| > M \end{cases}.$$

Find an upper bound for the variance of Y_i^M, and use this to upper bound the variance of $\bar{Y}_n^M$ (here $\bar{Y}_n^M$ is the sample mean of $Y_1^M, \ldots,$ defined similarly to $\bar{X}_n$). Specialize your two formulas for the variance to $\text{var}(Y_i^n)$ and $\text{var}(\bar{Y}_n^n)$ (i.e. these are truncated at value $M = n$). Prove that $\mathbf{P}(\bar{Y}_n > \epsilon) \to 0$ as $n \to \infty$.

(c) Argue that

$$\mathbf{P}(|\bar{X}_n| > \epsilon) \leq \mathbf{P}(\bar{X}_n \neq \bar{Y}_n^n) + \mathbf{P}(\bar{Y}_n^n > \epsilon).$$

(d) Note that the second term in the preceding equation was shown to approach zero in part (b). In this step we deal with the first term, and show that $\mathbf{P}(\bar{X}_n \neq \bar{Y}_n^n) \to 0$. First show that

$$\mathbf{P}(\bar{X}_n \neq \bar{Y}_n) \leq \sum_{i=1}^{n} \mathbf{P}(|X_i| > n) = n\mathbf{P}(|X_i| > n).$$

Next show that

$$n\mathbf{P}(|X_i| > n) \to 0.$$

(HINT: What if this was not true? What would it say about $\mathrm{E}\,|X|$?)