UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 7 Solutions
Spring 2011

Solution to Problem 1. We use the following notation:
B_i be the event that bulb chosen is from line i.
T is the random variable for the length of the lifetime of a bulb.

(a) Since the fast line is twice as fast as the other line, two of every three bulbs is produced by the first line. Thus,

$$P(B_1) = \frac{2}{3}$$

(b) By Bayes' Rule,

$$P(B_1|T > t) = \frac{P(T > t|B_1)P(B_1)}{P(T > t)},$$

where $P(T > t) = P(B_1)P(T > t|B_1) + P(B_2)P(T > t|B_2)$. Thus,

$$P(B_1|T > t) = \frac{\frac{2}{3}e^{-\lambda_1 t}}{\frac{2}{3}e^{-\lambda_1 t} + \frac{1}{3}e^{-\lambda_2 t}} = \frac{1}{1 + \frac{1}{2}e^{(\lambda_1-\lambda_2)t}}.$$

(c) We need to find a t such that $P(B_1|T > t) \leq \frac{1}{2}$. Thus,

$$P(B_1|T > t) = \frac{1}{1 + \frac{1}{2}e^{(\lambda_1-\lambda_2)t}} \leq \frac{1}{2}.$$

Inverting the fractions, while reversing the inequality,

$$1 + \frac{1}{2}e^{(\lambda_1-\lambda_2)t} \geq 2 \Longrightarrow t \geq \frac{\ln 2}{\lambda_1 - \lambda_2}$$

Thus, for any $t \geq \frac{\ln 2}{\lambda_1-\lambda_2}$, $P(B_1|T > t) \leq \frac{1}{2}$.

(d) The PDF of the maintence cost is the deviation from the value a; thus, let $X = |T - a|$. We begin by finding the CDF of X, by $F_X(x) = P(X \leq x) = P(|T - a| \leq x)$. We consider 3 different ranges of x in finding this CDF. If $x < 0$, then $P(|T - a| \leq x) = 0$. Now, we consider the case where $0 \leq x < a$.

$$\begin{aligned} P(|T - a| \leq x) &= P(-x + a \leq T \leq x + a) = \int_{a-x}^{a+x} \lambda_1 e^{-\lambda_1 t} dt \\ &= e^{-\lambda_1(a-x)} - e^{-\lambda_1(a+x)}. \end{aligned}$$

Lastly, if $a \le x$, then

$$\begin{aligned} P(|T-a| \le x) &= P(0 \le T \le x+a) = \int_0^{a+x} \lambda_1 e^{-\lambda_1 t} dt \\ &= 1 - e^{-\lambda_1 (a+x)}. \end{aligned}$$

Therefore, the CDF of X,

$$F_X(x) = \begin{cases} 0 & x < 0; \\ e^{-\lambda_1(a-x)} - e^{-\lambda_1(a+x)} & 0 \le x < a; \\ 1 - e^{-\lambda_1(a+x)} & a \le x; \end{cases}$$

Differentiating, we find the PDF of X,

$$f_X(x) = \begin{cases} 0 & x < 0; \\ \lambda_1 e^{-\lambda_1 a}(e^{\lambda_1 x} + e^{-\lambda_1 x}) & 0 \le x < a; \\ \lambda_1 e^{-\lambda_1 a} e^{-\lambda_1 x} & a \le x; \end{cases}$$

To find the $E[X]$, we integrate $f_X(x)$,

$$\begin{aligned} E[X] &= \int_0^\infty x f_X(x) dx \\ &= e^{-\lambda_1 a}\left[\int_0^a \lambda_1 x e^{-\lambda_1 x} dx + \int_0^a \lambda_1 x e^{\lambda_1 x} dx + \int_a^\infty \lambda_1 x e^{-\lambda_1 x} dx\right] \\ &= e^{-\lambda_1 a}\left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 a}}{\lambda_1} - a e^{-\lambda_1 a} + a e^{\lambda_1 a} - \frac{e^{-\lambda_1 a}}{\lambda_1} + \frac{1}{\lambda_1} + a e^{-\lambda_1 a} + \frac{e^{-\lambda_1 a}}{\lambda_1}\right] \\ &= \frac{2e^{-\lambda_1 a}}{\lambda_1} + a - \frac{1}{\lambda_1}, \end{aligned}$$

where integration by parts was needed.

Solution to Problem 2. (a) Using the information in the problem and moment generating properties $M_X(0) = 1$ and $\frac{d}{ds} M_X(s)\Big|_{s=0} = E[X]$, we obtain a system of two equations for a and b:

$$M_X(0) = ae^0 + be^{13(e^0-1)} = 1 \Rightarrow a + b = 1;$$
$$E[X] = 3 = ae^0 + 13be^0 e^{13(e^0-1)} \Rightarrow a + 13b = 5.$$

Therefore,

$$a = \frac{2}{3}, \quad b = \frac{1}{3}.$$

(b) $E[e^{5X}] = M_X(s)\Big|_{s=5} = ae^5 + be^{13(e^5-1)} = \frac{2}{3}e^5 + \frac{1}{3}e^{13(e^5-1)} = 6.20 \times 10^{831}.$

(c) One way to solve this part is to find the PDF of X and then find $P(X = 1)$.

We will demonstrate an alternative way of computing $P(X = 1)$ directly from the transform. This is useful if one cannot invert the transform and obtain the PDF directly.

It is easy to see that X is a discrete random variable (its transform is a combination of transforms of two PMFs). Moreover, X takes on nonnegative values (either 1 or the values of a Poisson random variable). There is a fact about the transform of a nonnegative discrete random variable which will be very useful here:

$$\frac{d^n}{d(e^s)^n} M_X(s)\bigg|_{e^s=0} = n! p_X(n).$$

To see this, note that for a nonnegative discrete random variable X we can write

$$M_X(s) = E[e^{sX}] = p_X(0) \cdot e^{0s} + p_X(1) \cdot e^{1s} + p_X(2) \cdot e^{2s} + ...$$

Then,

$$M_X(s)\bigg|_{e^s=0} = p_X(0),$$

$$\frac{d}{de^s} M_X(s)\bigg|_{e^s=0} = (p_X(1) \cdot 1 + 2p_X(2)e^s + ...)\bigg|_{e^s=0} = p_X(1),$$

$$\frac{d^2}{d(e^s)^2} M_X(s)\bigg|_{e^s=0} = (2!p_X(2) + 3!p_X(3)e^s + ...)\bigg|_{e^s=0} = 2!p_X(2),$$

etc. Therefore,

$$P(X = 1) = p_X(1) = \frac{1}{1!}\left(ae^s + be^{13(e^s-1)}\right)'\bigg|_{e^s=0} = \left(a + 13be^{13(e^s-1)}\right)\bigg|_{e^s=0} =$$

$$= a + 13be^{-13} = \frac{2}{3} + 13 \cdot \frac{1}{3} \cdot e^{-13} = 0.667.$$

(d) $E[X^2] = \frac{d^2}{ds^2} M(s)\bigg|_{s=0} = \left(ae^s + 13b(e^s e^{13(e^s-1)} + e^s \cdot 13e^s \cdot e^{13(e^s-1)})\right)\bigg|_{s=0}$

$$= a + 182b = \frac{184}{3}.$$

Solution to Problem 3. We have that $Z = E[X|P]$. Therefore we have:

$$E[Z] = E[E[X|P]] = E[X].$$

Now for the computation. First we find $E[X|P]$:

$$Z = E[X|P] \quad = \quad \frac{1}{1-P}$$

Now we proceed to calculate $E[Z]$:

$$\begin{aligned} E[Z] &= \int z\, dz \\ &= \int_0^{n-1/n} \frac{1}{1-p} \cdot \frac{n}{n-1} \\ &= \frac{n}{n-1} \cdot \ln n \end{aligned}$$

And therefore $\lim_{n\to\infty} E[Z] = \infty$ as it grows as $\log n$.

Solution to Problem 4. ...

(a) X, Y cannot be independent, since given X we know the value of Y to within two values, and hence it is easy to show that:

$$f(x|y) \neq f(x).$$

(b) Y, Z are independent because X is symmetric about around the origin (i.e. what we typically call the Y-axis).

(c)

$$\begin{aligned} f_{YZ}(y,z) &= f_{Y|Z}(y|z) \cdot f_Z(z) \\ &= f_X(x) \cdot f_Z(z) \end{aligned}$$

and therefore:

$$f_Y(y) = \sum_I f_X(x) \cdot f_Z(z) = f_X(x)$$

and therefore $Y \sim N[0,1]$ as desired.

(d) We want to show that $cov(X,Y) = 0$. Since $E[X] = E[Y] = 0$, we have:

$$\begin{aligned} cov(X,Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_{Y|X}(y|x) \cdot f_X(x)dxdy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_X(x)\frac{1}{2}(\delta(x) + \delta(-x) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} y[xf_X(x) - xf_X(x)] \\ &= 0 \end{aligned}$$

as required. The last equality follows from the fact that since X is a standard normal random variable, $f_X(x) = f_X(-x)$. Note that we have two dependent normal random variables X, Y that have zero correlation. There is a small subtlety here. We know that if two random variables have bivariate joint distribution, and are uncorrelated, then they are independent. However in this case, we have two dependent normal random variables, whose correlation is zero. The difference here is that the joint distribution is not bivariate normal.

Solution to Problem 5. The problem is simplified by looking at the fraction of the original stake that the gambler has at any given moment. Because the expected value operation is linear, we can compute the expected fraction of the original stake and multiply by the original stake to get the expected total fortune (the original stake is a constant).

If the gambler has a at the beginning of a round, he bets $a(2p-1)$ on the round. If he wins, he'll have $a+a(2p-1)$ units. If he loses, he'll have $a-a(2p-1)$ units. Thus at the end of the round, he will have $2pa$ following a win, and $2(1-p)a$ following a loss.

Thus, we see that winning multiplies the gambler's fortune by $2p$ and losing multiplies it by $2(1-p)$. Therefore, if he wins k times and loses m times, he will have $(2p)^k(2(1-p))^m$ times his original fortune. We can also compute the probability of this event. Let Y be the number of times the gambler wins in the first n gambles. Then Y has the binomial PMF:

$$p_Y(y) = \binom{n}{y} p^y (1-p)^{n-y}, \qquad y = 0, 1, \ldots, n.$$

We can now calculate the expected fraction of the original stake that he has after n gambles. Let Z be a random variable representing this fraction. We know that Z is related to Y via

$$Z = (2p)^Y (2(1-p))^{n-Y}.$$

We will calculate the expected value of Z using the PMF of Y.

$$\begin{aligned}
E[Z] &= \sum_{y=0}^{n} Z(y) p_Y(y) = \sum_{y=0}^{n} (2p)^y [2(1-p)]^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
&= \sum_{y=0}^{n} 2^y p^y 2^{n-y} (1-p)^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
&= 2^n \sum_{y=0}^{n} p^y (1-p)^{n-y} \binom{n}{y} p^y (1-p)^{n-y} \\
&= 2^n \sum_{y=0}^{n} \binom{n}{y} \left(p^2\right)^y \left[(1-p)^2\right]^{n-y} \\
&= 2^n \left(p^2 + (1-p)^2\right)^n,
\end{aligned}$$

where the last equality follows using the generalized binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

Thus the gambler's expected fortune is

$$2^n \left(p^2 + (1-p)^2\right)^n x,$$

where x is the fortune at the beginning of the first round.

An alternative method for solving the problem involves using iterated expectations. Let X_k be the fortune after the kth gamble. Again, we use the fact that the expected fortune after the kth gamble is

$$X_k = 2\left(p^2 + (1-p)^2\right) X_{k-1}.$$

Therefore, using iterated expectations, the fortune after n gambles is

$$\begin{aligned}
E[X_n] &= E\left[E[X_n|X_{n-1}]\right] \\
&= 2\left(p^2 + (1-p)^2\right) E[X_{n-1}] \\
&= 2\left(p^2 + (1-p)^2\right) E\left[E[X_{n-1}|X_{n-2}]\right] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^2 E[X_{n-2}] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^2 E\left[E[X_{n-2}|X_{n-3}]\right] \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^3 E[X_{n-3}] \\
&= \cdots \\
&= \left(2\left(p^2 + (1-p)^2\right)\right)^n E[X] \\
&= 2^n \left(p^2 + (1-p)^2\right)^n x.
\end{aligned}$$

Solution to Problem 6. Let A_t (respectively, B_t) be a Bernoulli random variable which is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_tB_t] = 0$ and $\mathbf{E}[A_tB_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = p_1p_2$ for $s \neq t$. We have

$$\mathbf{E}[X_1X_2] = \mathbf{E}\big[(A_1+\cdots+A_n)(B_1+\cdots+B_n)\big] = n\mathbf{E}\big[A_1(B_1+\cdots+B_n)\big] = n(n-1)p_1p_2,$$

and

$$\text{cov}(X_1, X_2) = \mathbf{E}[X_1X_2] - \mathbf{E}[X_1]E[X_2] = n(n-1)p_1p_2 - np_1np_2 = -np_1p_2.$$

Solution to Problem 7. ...

(a) Let N be the outcome of die. N is a discrete random variable taking values of 1,2, or 3 equally likely. So,

$$\begin{aligned}
E[N] &= \frac{1}{3}1 + \frac{1}{3}2 + \frac{1}{3}3 \\
&= 2 \\
Var(N) &= \frac{1}{3}(1-E[N])^2 + \frac{1}{3}(2-E[N])^2 + \frac{1}{3}(3-E[N])^2 \\
&= \frac{2}{3}
\end{aligned}$$

Let X be the result of spinning the wheel of fortune. It is a uniform random varible between $(0, 1)$. Thus:

$$\begin{aligned} E[X] &= \frac{1}{2} \\ Var(X) &= \frac{1}{12} \end{aligned}$$

$Y = X_1 + \cdots + X_N$ where N is the outcome of die. Using properties listed on pg. 234 of the text, we get:

$$\begin{aligned} E[Y] &= E[X]E[N] \\ &= 2\left(\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

(b) Similarly,

$$\begin{aligned} Var(Y) &= Var(X)E[N] + E[X]^2 Var(N) \\ &= \left(\frac{1}{12}\right)2 + \left(\frac{1}{2}\right)^2 \frac{2}{3} \\ &= \frac{1}{6} + \frac{2}{12} \\ &= \frac{1}{3} \end{aligned}$$