UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 6

Spring 2011

Issued: Tuesday, February 22, 2011 **Due:** Tuesday, March 8, 2011

Reading: Bertsekas & Tsitsiklis, Chap. 3 and §4.1

Problem 1. Random variable R has the PDF

$$f_R(r) = \begin{cases} \gamma r, & 1 < r \leq 3 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

where γ is some constant. Let A be the event $\{R \geq 2\}$.

(a) Find γ such that f_R is a valid PDF.

(b) Find the CDF of R.

(c) Calculate $\mathbb{E}(R)$, $\mathbb{P}(A)$, and $\mathbb{E}(R \mid A)$. Determine $f_{R|A}$.

(d) Let $W := R^2$. Find $\mathbb{E}(W)$ and $\text{var}(W)$.

(e) Let $Z := e^{-R}$. Find the PDF of Z.

(f) Let (X, Y) be uniformly distributed in the annulus

$$\left\{(x, y) \in \mathbb{R}^2 : \ a_0^2 \leq x^2 + y^2 \leq a_1^2\right\}$$

where $a_1 > a_0 > 0$ are constants. Someone claims that one can obtain a random variable R with PDF f_R given in (1), by setting $R := \sqrt{X^2 + Y^2}$ and choosing a_0 and a_1 appropriately. Argue in favor or against this claim.

Problem 2. Alice and Bob work independently on a problem set. The time for Alice to complete the set is exponentially distributed with mean 4 hours. The time for Bob to complete the set is exponentially distributed with mean 6 hours.

(a) What is the probability that it takes Alice more than 4 hours to finish?

(b) Given that Alice has been working on the problem set already for 4 hours, what is the probability that it takes her more than 8 hours to finish (i.e., four hours more)?

(c) What is the probability that Alice finishes the problem set before Bob?

(d) What is the probability that one of them finishes the problem set an hour or more before the other one?

Problem 3. Random variables X and Y are uniformly distributed in the region $0 \leq Y \leq X \leq 2$. Let $T := X + Y$.

(a) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(T)$.

(b) Would you expect the variance of T to be less than, equal to, or greater than the sum of the variances of X and Y? Calculations are not required, but you are to provide a lucid argument for your answer.

Problem 4. Let $X_1, X_2, \ldots, X_n$ be independent exponential random variables with rates $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. Find the PDF of $Y := \min\{X_1, X_2, \ldots, X_n\}$.

Problem 5. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $Z = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. Good and bad weather are equally likely. In the absence of any weather information:

(a) Calculate the PDF of Z.

(b) Calculate the probability that Z is between 1 and 3.

Problem 6. (Quantization) An analog-to-digital device converts a continuous signal into a discrete set of values. For example, the intensity in images are frequently converted to 8 bits, or $2^8 = 256$ levels of resolution, in a process known as quantization. As a particular example of a quantizer, consider the "greatest integer" function $\lfloor y \rfloor$, which maps the real number y to the largest integer $\leq y$, so that for example $\lfloor 3.85 \rfloor = 3$. Supposing that X is exponentially distributed with rate $\lambda = 1$, compute $\mathbb{E}(\lfloor X \rfloor)$.

Problem 7. Let $X, X_1, X_2, \ldots, X_n$ be i.i.d. $N(0, \sigma^2)$ random variables. (That is, independent zero-mean Gaussian RVs with variance σ^2.)

(a) Derive an expression for $\mathbb{E}(X^n)$, the nth moment of X, for all $n \in \mathbb{N}$.

[*Hint*: Consider the cases of n even and n odd, separately. Relate the nth moment to the $(n-2)$th moment.]

(b) Find the mean and the variance of $Y := X_1^2 + X_2^2 + \cdots + X_n^2$.

(c) Find the mean and the variance of $W := |X_1| + |X_2| + \cdots + |X_n|$.

(d) Find an upper bound on $\mathbb{E}(\sqrt{Y})$ where Y is as in part (b).

[*Hint:* It might help to note that $x \mapsto -\sqrt{x}$ is a convex function on $x \geq 0$.]

(e) Find the PDF of Y, assuming $\sigma = 1$ and $n = 2$. [*Bonus:* do it for general n.]

Problem 8. The Beta distribution is a two-parameter continuous distribution, denoted as $\mathcal{B}e(\alpha,\beta)$ (for $\alpha,\beta>0$) with PDF

$$f(x)=\begin{cases}\frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}, & x\in(0,1)\\ 0, & \text{otherwise}\end{cases}$$

where $B(\alpha,\beta):=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx$ is the *Beta function.* Note that $\mathcal{B}e(1,1)$ is just uniform distribution on $(0,1)$. It is a popular choice for modeling nonuniform distributions on $(0,1)$ by playing with the two parameters. The mean of $\mathcal{B}e(\alpha,\beta)$ is $\frac{\alpha}{\alpha+\beta}$. Now, consider a coin which has (unknown) probability θ of coming up heads in each throw. Let X denote the number of heads in tossing the coin n times.

(a) Oscar with no prior belief on θ, models it as a uniform random variable on $(0,1)$. Given that he observes $X=x$ heads in n throws, obtain the posterior distribution and mean of θ as a function of x and n. Evaluate you results for $x=3$ and $n=10$.

(b) Stephan with prior belief that the coin is extremely unfair, models θ as a $\mathcal{B}e(\frac{1}{2},\frac{1}{2})$ distribution. Answer the questions of part (a) for this new setup.

(c) Cindy with prior belief that θ is roughly either $1/3$ or $2/3$ due to natural irregularities models θ as follows: θ is distributed as $\mathcal{B}e(10,20)$ if the coin is old and as $\mathcal{B}e(20,10)$ if the coin is new. A coin is old or new with equal probability. Answer the questions of part (a) for this new setup.

[*Bonus:* if you plot the prior versus posterior distributions in each cases and repeat the problem for $n=100$ and $x=36$ and provide an explanation for your results.]