UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 5
Spring 2011

Issued: Tuesday, February 15, 2011 **Due:** In HW box Tuesday, Feb 22, 2011

Reading: Bertsekas & Tsitsiklis, §2.5–2.7 and §3.1

Problem 1. Constantinos will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops. We are told that

$$p_{N|K}(n \mid k) = \frac{1}{k}, \qquad \text{for } n = 1, \ldots, k.$$

(a) Find the joint PMF of K and N.

(b) Find the marginal PMF of N.

(c) Find the conditional PMF of K given that $N = 2$.

(d) We are now told that he bought at least 2 but no more than 3 books. Find the conditional mean and variance of K, given this piece of information.

(e) The cost of each book is a random variable with mean 3. What is the expected value of his total expenditure? *Hint:* Condition on events $N = 1, \ldots, N = 4$ and use the total expectation theorem.

Problem 2. Oscar has a donut eating problem and needs your help. At his workplace, the first thing Oscar does every morning is to go to the kitchen and grab either one, two, or three donuts, with each of these possibilities being equally likely. If he takes three donuts, he does not return to the supply room again that day (you probably wouldn't either, these are pretty big donuts). If he receives only one or two donuts, he will later become hungry and make one additional trip to the kitchen, where he again eats one, two, or three donuts, with equal probability.
Note: The number of donuts eaten in one trip will not affect the number of donuts eaten in any other trip. Evaluate:

(a) $\mathbf{P}(A)$, where A is the event that Oscar eats a total of three donuts on a given day.

(b) $\mathbf{P}(B|A)$, where B is the event that he visited the kitchen twice on the day in question.

(c) $\mathrm{E}[N]$ and $\mathrm{E}[N|C]$, where N is the total number of donuts Oscar eats on any given day, and C is the event that $(N > 3)$).

(d) $\sigma_{N|C}$, the conditional standard deviation of N given C. (Find the conditional variance and take the square root.)

(e) $\mathbf{P}(D)$, where D is the event that he eats more than three donuts on *each* of the next 16 days.

Problem 3. $X_1, \cdots, X_n$ be independent random variables

(a) Let $X_{min} = \min(X_1, \cdots, X_n)$ and $X_{max} = \max(X_1, \cdots, X_n)$. Show that

$$\begin{aligned} P(X_{min} \geq x) &= \prod_{i=1}^{n} P(X_i \geq x) \\ P(X_{max} < x) &= \prod_{i=1}^{n} P(X_i < x). \end{aligned}$$

(b) Let $g_1 = g_1(X_1, \cdots, X_k)$ and $g_2 = g2(X_{k+1}, \cdots, X_n)$, functions respectively of $(X_1, \cdots, X_k)$ and $(X_{k+1}, \cdots, X_n)$. Show that the random variables g_1 and g_2 are independent.

Problem 4. Computer chips can be expected to fail after operating for a random amount of time. Suppose, in particular, that

$$\mathbf{P}(\text{chip still works at time } t) = e^{-\alpha t}, \quad t \geq 0. \tag{1}$$

Consider now that we have a manufacturing process that produces a mix of "good" and "bad" chips. The lifetime of good chips satisfies Eq. (1). The lifetime of bad chips satisfies the same relation except that α is replaced by 1000α. Assume that the fraction of good chips is p and the fraction of bad chips $1 - p$.

(a) Find the probability that a randomly selected chip is still functioning after τ time units of operation.

(b) In order to weed out bad chips, every chip is tested for τ time units before leaving the factory, and only chips that do not fail during the testing period are shipped to customers. Give a formula for the probability that a customer receives a bad chip (as a function of the constants α, p, and τ).

Problem 5. A pair of fair four-sided dice is thrown once. Each die has faces labeled 1, 2, 3, and 4. Discrete random variable X is defined to be the product of the down-face values. Determine the conditional variance of X^2 given that the sum of the down-face values is greater than the product of the down-face values.

Problem 6. Consider the two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ (1-p)\lambda e^{\lambda x}, & \text{if } x < 0 \end{cases}$$

where λ and p are real numbers with $\lambda > 0$ and $p \in [0, 1]$. Find the mean and the variance of X.

Problem 7. Recall that one of the axioms of probability is if $A_1, A_2, ...$ is a countable number of disjoint subsets (events) of the sample space:

$$P(\cup_i A_i) = \sum_i P(A_i).$$

There are strange subsets for which this axiom is not possible to apply. In this problem we will construct one such set.
Let $\Omega = [0,1]$. If a random variable X is distributed uniformly over Ω: $P(a < x < b) = 1/(b-a)$. But suppose we want to allow **all** possible subsets of $[0,1]$ (not just intervals). Then we can construct one for which there can be no definition of "length":
Define any two $x, y \in [0, 1/3]$ to be "equivalent numbers" of the same class if $x - y$ is a rational number. The set S is such that exactly one number is included from each equivalence class.

(a) Show that for any two numbers $p, q \in S$, $p - q$ is not a rational number.

(b) Argue that for any number $u \in [0, 1/3]$, there is exactly one number equivalent to u in S.

(c) Define the set C to be all the rational numbers in $[0, 2/3]$, and for any $x \in C$ define S_x as the set obtained by adding x to each element of S. Show that for any two $x, y \in C$,

$$S_x \cap S_y = \varnothing\,.$$

(d) Pick any number $w \in [1/3, 2/3]$. Show that there must exist $x \in C$ such that $w \in S_x$. Hint: First argue $w - 1/3$ must be equivalent to some member of S.

(e) From the previous part, argue that $\cup_{x \in C} S_x$ contains $[1/3, 2/3]$.

(f) Use axioms of probability to show that

$$1 \geq \sum_{x \in C} P(S_x) \geq P([1/3, 2/3]) = 1/3$$

(g) Since for any $x \in C$, S_x is a shifted version of S, the two sets (S and S_x) must have the same length, L. From this and the previous part argue that either choice: $L = 0$ or $L > 0$ violates axioms of probability.