UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 5 Solutions
Spring 2011

Solution to Problem 1. We are given the following information:

$$p_K(k) = \begin{cases} 1/4, & \text{if } k = 1,2,3,4; \\ 0, & \text{otherwise} \end{cases}$$

$$p_{N|K}(n \mid k) = \begin{cases} 1/k, & \text{if } n = 1,\dots,k; \\ 0, & \text{otherwise} \end{cases}$$

(a) We use the fact that $p_{N,K}(n,k) = p_{N|K}(n \mid k)p_K(k)$ to arrive at the following joint PMF:

$$p_{N,K}(n,k) = \begin{cases} 1/(4k), & \text{if } k = 1,2,3,4 \text{ and } n = 1,\dots,k; \\ 0, & \text{otherwise} \end{cases}$$

(b) The marginal PMF $p_N(n)$ is given by the following formula:

$$p_N(n) = \sum_k p_{N,K}(n,k) = \sum_{k=n}^{4} \frac{1}{4k}$$

On simplification this yields

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48, & n = 1; \\ 1/8 + 1/12 + 1/16 = 13/48, & n = 2; \\ 1/12 + 1/16 = 7/48, & n = 3; \\ 1/16 = 3/48, & n = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(c) The conditional PMF is

$$p_{K|N}(k \mid 2) = \frac{p_{N,K}(2,k)}{p_N(2)} = \begin{cases} 6/13, & k = 2; \\ 4/13, & k = 3; \\ 3/13, & k = 4; \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let A be the event $2 \leq N \leq 3$. We first find the conditional PMF of K given

A.

$$\begin{aligned}
p_{K|A}(k) &= \frac{\mathbf{P}(K=k,A)}{\mathbf{P}(A)} \\
\mathbf{P}(A) &= p_N(2)+p_N(3) = \frac{5}{12} \\
\mathbf{P}(K=k,A) &= \begin{cases} \frac{1}{8}, & k=2; \\ \frac{1}{12}+\frac{1}{12}, & k=3; \\ \frac{1}{16}+\frac{1}{16}, & k=4; \\ 0, & \text{otherwise} \end{cases} \\
p_{K|A}(k) &= \begin{cases} \frac{3}{10}, & k=2; \\ \frac{2}{5}, & k=3; \\ \frac{3}{10}, & k=4; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

Because the conditional PMF of K given A is symmetric around $k=3$, we know $\boxed{\mathbf{E}[K \mid A]=3}$. We now find the conditional variance of K given A.

$$\begin{aligned}
var(K \mid A) &= \mathbf{E}[(K-\mathbf{E}[K \mid A])^2 \mid A] \\
&= \frac{3}{10}\cdot(2-3)^2+\frac{2}{5}\cdot 0+\frac{3}{10}\cdot(4-3)^2 \\
&= \boxed{\tfrac{3}{5}}
\end{aligned}$$

(e) Let C_i be the cost of book i and $\mathbf{E}[C_i]=3$. Let T be the total cost, so $T=C_1+\ldots+C_N$. We now find $\mathbf{E}[T]$ using the total expectation theorem.

$$\begin{aligned}
\mathbf{E}[T] &= \mathbf{E}[T \mid N=1]p_N(1)+\mathbf{E}[T \mid N=2]p_N(2)+\mathbf{E}[T \mid N=3]p_N(3)+\mathbf{E}[T \mid N=4]p_N(4) \\
&= \mathbf{E}[C_1]p_N(1)+\mathbf{E}[C_1+C_2]p_N(2)+\mathbf{E}[C_1+C_2+C_3]p_N(3)+\mathbf{E}[C_1+C_2+C_3+C_4]p_N(4) \\
&= \mathbf{E}[C_i]p_N(1)+2\mathbf{E}[C_i]p_N(2)+3\mathbf{E}[C_i]p_N(3)+4\mathbf{E}[C_i]p_N(4) \\
&= 3\cdot\frac{25}{48}+6\cdot\frac{13}{48}+9\cdot\frac{7}{48}+12\cdot\frac{1}{16} \\
&= \boxed{\tfrac{21}{4}}
\end{aligned}$$

Solution to Problem 2. (a) Let D_1 be the number of RV that Oscar eats on the first trip, and D_2 be the number of donuts he eats on the second trip. Then A, the event that Oscar eats 3 donuts in a day is given by the mutually exclusive events

$$A=\{D_1=3\}\cup\{D_1=1,D_2=2\}\cup\{D_1=2,D_2=1\}.$$

Thus $\mathbf{P}(A)=\mathbf{P}(D_1=3)+\mathbf{P}(D_1=1,D_2=2)+\mathbf{P}(D_1=2,D_2=1)=\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3}=\frac{5}{9}$.

(b)

$$\mathbf{P}(B|A)=\frac{\mathbf{P}(B\cap A)}{\mathbf{P}(A)}=\frac{2/9}{5/9}=\frac{2}{5}.$$

This follows from (a) and the fact that $\{B\cap A\}=\{D_1=1,D_2=2\}\cup\{D_1=2,D_2=1\}$.

(c) We note that

$$\begin{aligned}
\mathbf{P}(N=2) &= \mathbf{P}(D_1=1, D_2=1) = \frac{1}{9} \\
\mathbf{P}(N=3) &= \mathbf{P}(A) = \frac{5}{9} \\
\mathbf{P}(N=4) &= \mathbf{P}(D_1=1, D_2=3) + \mathbf{P}(D_1=2, D_2=2) = \frac{2}{9} \\
\mathbf{P}(N=5) &= \mathbf{P}(D_1=2, D_2 03) = \frac{1}{9}
\end{aligned}$$

We compute $\mathbf{E}[N]$:

$$\mathbf{E}[N] = \sum_n n p_N(n) = 2\cdot\frac{1}{9} + 3\cdot\frac{5}{9} + 4\cdot\frac{2}{9} + 5\cdot\frac{1}{9} = \frac{10}{3}$$

The conditional PMF $p_{N|C}(n)$ is given by

$$p_{N|C}(n) = \begin{cases} 2/3, & n=4; \\ 1/3, & n=5; \\ 0, & \text{otherwise} \end{cases}$$

so

$$\mathbf{E}[N|C] = \sum_n n p_{N|C}(n) = 4\cdot\frac{2}{3} + 5\cdot\frac{1}{3} = \frac{13}{3}.$$

(d) Using the conditional PMF from the previous part, one can compute

$$\mathbf{E}[N^2|C] = \sum_n n^2 p_{N|C}(n) = 4^2\cdot\frac{2}{3} + 5^2\cdot\frac{1}{3} = \frac{57}{3}.$$

Then,

$$\sigma^2_{N|C} = \mathbf{E}[N^2|C] - (\mathbf{E}[N|C])^2 = \frac{57}{3} - \frac{169}{9} = \frac{2}{9}$$

so $\sigma_{N|C} = \frac{\sqrt{2}}{3}$.

(e) The probability that Oscar eats more than three Donuts on any one day is

$$p_N(4) + p_N(5) = \frac{1}{3}.$$

Since all actions on separate days are independent, $\mathbf{P}(D) = \left(\frac{1}{3}\right)^{16}$.

Solution to Problem 3. ...

(a) If $X_{min} \geq x$, all $X_1, \cdots, X_n$ must be $\geq x$. Since the X_i are independent, the probability that all of them are $\geq x$ is the product of the probabilities that each $X_i \geq x$. Therefore,

$$P(X_{min} \geq x) = \prod_{i=1}^{n} P(X_i \geq x).$$

Similarly, if $X_{max} < x$, all $X_1, \cdots, X_n$ must be $< x$, and

$$P(X_{max} < x) = \prod_{i=1}^{n} P(X_i < x).$$

(b) Let A and B be some arbitrary subsets of the sample space. Our usual definition of independence between some random variables X and Y is that $f_{XY}(x,y) = f_X(x)f_Y(y)$. However, this is equivalent to saying that $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ (the probability of having X in set A and Y in set B is the product of the probabilities of having X in A and Y in B for any A and B if and only if X and Y are independent). We also consider that for some $f(x) \in A$, $x \in f^{-1}(A)$ (sometimes called the "pre-image" of A). We know that $X_1, \cdots, X_k$ is independent of $X_{k+1}, \cdots, X_n$. So,

$$\begin{aligned} P(g_1 \in A, g_2 \in B) &= P(f_1(X_1, \cdots, X_k) \in A, f_2(X_{k+1}, \cdots, X_n) \in B) \\ &= P(X_1, \cdots, X_k \in f_1^{-1}(A), X_{k+1}, \cdots, X_n \in f_2^{-1}(B)) \\ &= P(X_1, \cdots, X_k \in f_1^{-1}(A))P(X_{k+1}, \cdots, X_n \in f_2^{-1}(B)) \\ &= P(f_1(X_1, \cdots, X_k) \in A)P(f_2(X_{k+1}, \cdots, X_n) \in B) \\ &= P(g_1 \in A)P(g_2 \in B) \end{aligned}$$

Therefore g_1 and g_2 are independent.

Solution to Problem 4. Given the following events,
A_t: Chip still works at time t.
B: The chip is bad.
G: The chip is good.
We can specify that $\mathbf{P}(A_t|G) = e^{-\alpha t}$ and $\mathbf{P}(A_t|B) = e^{-1000\alpha t}$.

(a) By using the definition of conditional probability and the total probability theorem, we can solve:

$$\mathbf{P}(A_t) = \mathbf{P}(G)\mathbf{P}(A_t|G) + \mathbf{P}(B)\mathbf{P}(A_t|B) = pe^{-\alpha t} + (1-p)e^{-1000\alpha t}$$

(b) By using the definition of conditional probability, we get:

$$\mathbf{P}(B|A_t) = \frac{\mathbf{P}(B \cap A_t)}{\mathbf{P}(A_t)}.$$

Furthermore, $\mathbf{P}(B \cap A_t) = \mathbf{P}(B)\mathbf{P}(A_t|B) = (1-p)e^{-1000\alpha t}$. Therefore,

$$\mathbf{P}(B|A_t) = \frac{(1-p)e^{-1000\alpha t}}{pe^{-\alpha t} + (1-p)e^{-1000\alpha t}}.$$

Solution to Problem 5. ...
All possible outcomes are:
(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4),
(3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), and (4, 4)
Given that the sum of the down-face values is greater than the product of the down-face values, our universe is restricted to:
(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (3, 1), and (4, 1)
Then we have that:

$$p_X(x) = \begin{cases} \frac{1}{7} & \text{if } x = 1 \\ \frac{2}{7} & \text{if } x = 2 \\ \frac{2}{7} & \text{if } x = 3 \\ \frac{2}{7} & \text{if } x = 4 \\ 0 & \text{otherwise} \end{cases}$$

Let $Z = X^2$. Then we have:

$$p_Z(z) = \begin{cases} \frac{1}{7} & \text{if } z = 1 \\ \frac{2}{7} & \text{if } z = 4 \\ \frac{2}{7} & \text{if } z = 9 \\ \frac{2}{7} & \text{if } z = 16 \\ 0 & \text{otherwise} \end{cases}$$

$E[Z] = \frac{1}{7}(1) + \frac{2}{7}(4) + \frac{2}{7}(9) + \frac{2}{7}(16) = \frac{59}{7}$.
$Var(Z) = \frac{1}{7}(\frac{52}{7})^2 + \frac{2}{7}(\frac{31}{7})^2 + \frac{2}{7}(\frac{-4}{7})^2 + \frac{2}{7}(\frac{-53}{7})^2 = \frac{1468}{49} \approx 29.96$.

Solution to Problem 6. We have the two-sided exponential PDF

$$f_X(x) = \begin{cases} p\lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ (1-p)\lambda e^{\lambda x}, & \text{if } x < 0 \end{cases}.$$

Here we are trying to find $\mathbf{E}X$ and $\text{Var}(X)$. You can do this by splitting up the integral to $\int_0^\infty$ and $\int_{-\infty}^0$ and doing integration by parts. However, there is a cute way to do it. Let A be the event that $X > 0$. Using the total expectation theorem, we have

$$\begin{aligned} \mathbf{E}X &= \mathbf{E}(X|A)\mathbf{P}(A) + \mathbf{E}(X|A^c)\mathbf{P}(A^c) \\ &= p\frac{1}{\lambda} + (1-p)(-\frac{1}{\lambda}) \\ &= \frac{1}{\lambda}(2p-1)\,, \end{aligned}$$

where get $\mathbf{E}(X|A)$ and $\mathbf{E}(X|A^c)$ from the fact that each of the sides of our two sided exponential is an exponential pdf on its own.
Now, for the variance we use

$$\text{Var}X = \mathbf{E}(X^2) - (\mathbf{E}X)^2 = \mathbf{P}(A)\mathbf{E}(X^2|A) + \mathbf{P}(A^c)\mathbf{E}(X^2|A^c) - (\mathbf{E}X)^2\,.$$

For the first terms we have

$$\begin{aligned} \text{Var}(X|A) &= \mathbf{E}(X^2|A) - (\mathbf{E}[X|A])^2 \\ \frac{1}{\lambda^2} &= \mathbf{E}(X^2|A) - \frac{1}{\lambda^2} \\ \Rightarrow \mathbf{E}(X^2|A) &= \frac{2}{\lambda^2}\,. \end{aligned}$$

We can find $\mathbf{E}(X^2|A^c)$ in the same way. Thus

$$\text{Var}X = \frac{2p}{\lambda^2} + \frac{(1-p)2}{\lambda^2} - \frac{1}{\lambda^2}(2p-1)^2 = \frac{2}{\lambda^2} - \frac{(2p-1)^2}{\lambda^2}\,.$$

Solution to Problem 7. (a) Any two number $p, q \in S$ must belong to two different equivalent classes and therefore their difference cannot be rational.

(b) The number u must belong to exactly one equivalence class. Since S has one representative from every equivalence class, it must have one from the class u belongs to.

(c) The elements of S_x are of the form $p + x$ where $p \in S$, while those in S_y are of the form $q + y$ where $q \in S$. If $p + x = q + y$ for $(x \neq y)$ then $p - q = y - x$, i.e. $p - q$ must be rational. But that cannot be from part (a) since $p, q \in S$.

(d) For any $w \in [1/3, 2/3]$ is true that $w - 1/3 \in [0, 1/3]$. From part (b), there is exactly one element of S that is equivalent to $w - 1/3$. Let this number be w'. Then $w' - (w - 1/3) = 1/3 + w' - w$ is rational, and therefore $0 \leq w - w' \leq 2/3$ is rational. Thus $w - w' \in C$ and $w \in S_{w-w'}$.

(e) If there were a number $w \in [1/3, 2/3]$ but $w \notin \cup_{x \in C} S_x$, then $w - 1/3$ would not be equivalent to any element of S. But that contradicts (d).

(f) Since the S_x are disjoint, $P(\cup_{x \in C} S_x) = \sum_{x \in C} P(S_x)$. Next, by part (e), $\cup_{x \in C} S_x$ contains $[1/3, 2/3]$ so $\sum_{x \in C} P(S_x) \geq P[1/3, 2/3] = 1/3$. And $P(\cup_{x \in C} S_x) \leq 1$ since the intervals are disjoint and the probability of the union of any number of disjoint sets cannot exceed 1.

(g) $L = 0$ won't work since $\sum_{x \in C} P(S_x) \geq 1/3$ would yield $0 \geq 1/3$ and any $L > 0$ would yield $1 \geq \infty$ since the summation is over a countably infinite number of terms that have value L.