UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 4

Spring 2011

Issued: Tues, Feb 8, 2011 **Due:** In HW box (Cory 240) Tues, Feb 15, 2011

Reading: Bertsekas & Tsitsiklis, §2.1-2.4

Problem 1. Two four-sided dice are rolled simultaneously.

(a) Let X be the sum of the two rolls. Calculate the PMF and the expected value of X.

(b) Someone proposes to give you in dollars twice the amount of the sum X that you roll, if you pay A dollars in advance. What should be the amount A in order for you to expect to break even?

(c) Let $Y = -X^2 + 10X - 16$. Calculate the PMF and expected value of Y.

(d) Repeat part (b) using Y instead of X.

Problem 2. Answer each of the following questions by coming up with appropriate random variables and using properties of expectation.

(a) A permutation on the numbers $[n] = \{1, \ldots, n\}$ can be represented as a function $\pi : [n] \to [n]$, where $\pi(i)$ is the position of i in the ordering given by the permutation. A fixed point of a permutation $\pi : [n] \to [n]$ is a value for which $\pi(x) = x$. Find the expected number of fixed points of a permutation chosen uniformly at random from all permutations.

(b) (*Extra credit.*) We roll a standard fair die over and over. What is the expected number of rolls until the first pair of consecutive sixes appears? (*Hint:* The answer is not 36.)

(c) A standard deck of 52 cards has 13 hearts. The cards in such a deck are shuffed, and the top five cards are dealt to a player. What is the expected number of hearts that the player receives?

Problem 3. Consider the random variable X with PMF

$$p_X(x) = \begin{cases} \frac{x^2}{a} & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find a and $\mathbf{E}[X]$.

(b) What is the PMF of the random variable $Z = (X - \mathbf{E}[X])^2$?

(c) Using part (b) compute the variance of X.

Problem 4. A group of UC Berkeley entrepreneurs at Golden Bear Airlines just purchased troubled Air Stanford. Air Stanford currently only offers service to Reykjavik and Auckland. Because they are so unorganized, flights occur in a random manner and their planes often crash. The probability that a flight to Reykjavik or Auckland crashes is 1/5 and 1/10, respectively (given that your flight is to Auckland, it has probability 1/10 of crashing, *independent* of other crashes and similarly for Reykjavik). Any particular flight goes to Reykjavik with probability 2/3 and Auckland with probability 1/3.

(a) What is the probability that a randomly chosen flight crashes?

(b) What is the expected number of flights that occur before the first crash?

(c) What is the expected number of flights that occur after 3 non-crash flights before a crash? That is, suppose you've had three flights which didn't crash, for example the recent flight history is "crash, crash, safe, crash, safe, safe, safe". Starting now, what is the expected time until a crash?

(d) Air Stanford has 1000 flights per year. What is the probability that they have 1 or less crash in a year?

(e) Haas school graduates at Golden Bear Airlines discovered that if 100 new mechanics are hired, the probability of a safe flight on any particular Air Stanford flight will be 0.9999. Using the Poisson approximation, what is the probability that all 1000 flights in a given year arrive safely at their destination?

Problem 5. A particular circuit works if all ten of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either *ordinary* devices or *ultra-reliable* devices. An ordinary device has a failure probability of $q = 0.1$ while an ultra-reliable device has a failure probability of $q/2 = 0.05$, independent of any other device. However, each ordinary device costs \$1 whereas an ultra-reliable device costs \$3.

(a) Should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit $E[R]$? Keep in mind that your answer will depend on k.

(b) Suppose you can build a circiut using a mix of ordinary and ultra-reliable devices, say a is the number of ordinary devices. Would you ever want to use a mix of devices in each circuit, i.e. $0 < a < 10$? Again your goal is to maximize your expected profit $E[R]$.

Problem 6 (Poisson random variables). The following questions ask you to investigate some of the nice properties of Poisson random variables.

(a) Let X and Y be two independent Poisson random variables with means λ_1, λ_2, respectively. Prove that the sum, $Z = X + Y$, is a Poisson random variable (*Hint:* You may want to make use of the binomial theorem, $(\alpha + \beta)^n = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i}$). What is the pmf of Z?

(b) Let X be a Poisson random variable with mean μ, representing the number of errors in the notes for one EE126 lecture. Each error is independently a grammatical error with probability p and a spelling error with probability $1-p$. If Y and Z are random variables representing the number of grammatical and spelling errors, respectively, in one set of lecture notes, prove that Y and Z are Poisson random variables with means μp and $\mu(1-p)$, respectively. Also, prove that Y and Z are independent.

Problem 7. A function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* iff $f(\lambda_1 x + \lambda_2 y) \leq \lambda_1 f(x) + \lambda_2 f(y)$ for all $x, y \in \mathbb{R}$, $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq}$ with $\lambda_1 + \lambda_2 = 1$.

(a) Show by induction that if $f : \mathbb{R} \to \mathbb{R}$ is convex then for any $x_1, \ldots, x_n \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{\geq}$ with $\sum_{i=1}^{n} \lambda_i = 1$,

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i) .$$

(b) Use part (a) to prove that if X is a discrete random variable *with finite support* and $f : \mathbb{R} \to \mathbb{R}$ is convex then

$$\mathrm{E}[f(X)] \geq f(\mathrm{E}[X]) .$$

This is a very useful inequality known as Jensen's inequality.

(c) If X and Y are positive, independent and identically distributed, show that

$$\mathbf{E}\left[\frac{X}{Y}\right] \geq 1 .$$

(This statement may seem paradoxical: X and Y are identically distributed and therefore, in some sense, of the "same size." But, the result $\mathbf{E}[X/Y] \geq 1$ suggests that X tends to be "larger" than Y; note also that $\mathbf{E}[Y/X] \geq 1$ follows from your proof!)
Hint: If you are attempting this problem before Thursday, you will need the following result. If two random variables A and B are independent, then $\mathrm{E}[AB] = \mathrm{E}[A]E[B]$. Also, $p_{AB}(a, b) = p_A(a) p_B(b)$.