UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 4 Solutions
Spring 2011

Solution to Problem 1. ...
The sample space Ω for the experiment of rolling 2 four-sided dice is given by

$$\Omega = \{(1,1),(1,2),...,(4,4)\}.$$

As the outcomes of the 2 rolls are independent of each other and all 4 numbers are equally likely to come up, the probability of each element in Ω is $\frac{1}{16}$.

(a) Let $A_i = \{\omega \in \Omega | X(\omega) = i\}$.

Then,

$$\begin{aligned} A_2 &= \{(1,1)\} \\ A_3 &= \{(1,2),(2,1)\} \\ A_4 &= \{(1,3),(3,1),(2,2)\} \\ A_5 &= \{(1,4),(2,3),(3,2),(4,1)\} \\ A_6 &= \{(2,6),(3,3),(4,2)\} \\ A_7 &= \{(3,4),(4,3)\} \\ A_8 &= \{(4,4)\} \\ A_i &= \phi \text{ for } i \neq 2,3,4,5,6,7,8. \end{aligned}$$

The PMF of X, $p_X(i) = \mathbf{P}(A_i)$, is therefore given by

$$p_X(i) = \begin{cases} \frac{1}{16} & i = 2, \\ \frac{2}{16} & i = 3, \\ \frac{3}{16} & i = 4, \\ \frac{4}{16} & i = 5, \\ \frac{3}{16} & i = 6, \\ \frac{2}{16} & i = 7, \\ \frac{1}{16} & i = 8. \end{cases}$$

From symmetry, the expected value is 5. Using the formula for expected value, we can verify this as follows:

$$E[X] = \sum_{i=2}^{8} i p_X(i) = 2 \cdot \frac{1}{16} + 3 \cdot \frac{2}{16} + 4 \cdot \frac{3}{16} + 5 \cdot \frac{4}{16} + 6 \cdot \frac{3}{16} + 7 \cdot \frac{2}{16} + 8 \cdot \frac{1}{16} = 5$$

(b) Let P be your profit. Then $P = 2X - A$. We have

$$E[P] = E[2X - A] = 2E[X] - A$$

In order to break even on the average, pick A such that $E[P] = 0$. Hence, $A = 2E[X] = 10$.

(c) Let $B_k = \{\omega \in \Omega | Y(\omega) = k\}$

Then

$$\begin{aligned} B_0 &= \{\omega \in \Omega | X(\omega)^2 + 10X(\omega) - 16 = 0\} \\ &= \{(1,1),(4,4)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} B_5 &= \{(1,2),(2,1),(3,4),(4,3)\} \\ B_8 &= \{(1,3),(3,1),(2,2),(2,4),(3,3),(4,2)\} \\ B_9 &= \{(1,4),(2,3),(3,2),(4,1)\} \\ B_k &= \phi \text{ for } k \neq 0,5,8,9. \end{aligned}$$

The PMF of Y, $p_Y(k) = \mathbf{P}(B_k)$, is therefore given by

$$p_Y(k) = \begin{cases} \frac{2}{16} & k = 0, \\ \frac{4}{16} & k = 5, \\ \frac{6}{16} & k = 8, \\ \frac{4}{16} & k = 9. \end{cases}$$

Using the formula for expected value, we have:

$$E[Y] = 0 \cdot \frac{2}{16} + 5 \cdot \frac{4}{16} + 8 \cdot \frac{6}{16} + 9 \cdot \frac{4}{16} = 6\frac{1}{2}.$$

(d) Using the result from part (b), we still need $A = 2E[Y]$ for the expected profit to be zero. In this case, $A = 13$.

Solution to Problem 2. ...

(a) Let the random variable X denote the number of fixed points. Define the collection of indicator random variables $\{X_i, 1 \leq i \leq n\}$ by $X_i = 1_{\{\pi(i)=i\}}$. Then the number of fixed points is given by $X = \sum_{i=1}^{n} X_i$, and $\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}X_i$ by linearity of expectation. But

$$\mathbf{E}[X_i] = \mathbf{P}[X_i = 1] = \mathbf{P}[\pi(i) = i] = \frac{1}{n},$$

so $\mathbf{E}[X] = n \cdot \frac{1}{n} = 1$.

(b) Let A_1 denote the event that you roll a six on the first roll, and A_2 denote the event that you roll a six on the second roll. Note that $\mathbf{P}(A_i) = 1/6$ for $i = 1, 2$. Then if R denotes the time until you roll double sixes, we have by total expectation that

$$\mathbf{E}R = \mathbf{E}(R|A_1)\mathbf{P}(A_1) + \mathbf{E}(R|A_1^c)\mathbf{P}(A_1^c) = \frac{1}{6} \cdot \mathbf{E}(R|A_1) + \frac{5}{6} \cdot (1 + \mathbf{E}R)\,,$$

where we used the fact that $\mathbf{E}(R|A_1^c) = 1 + \mathbf{E}R$ since the process is memoryless. Now,

$$\mathbf{E}(R|A_1) = \mathbf{E}(R|A_1, A_2)\mathbf{P}(A_2|A_1) + \mathbf{E}(R|A_1, A_2^c)\mathbf{P}(A_2^c|A_1) = \frac{1}{6} \cdot 2 + \frac{5}{6} \cdot (2 + \mathbf{E}R)\,,$$

where we used the independence of A_1, A_2 and the fact that after a non-six roll the count starts over. Hence

$$\mathbf{E}R = \frac{1}{6^2} \cdot 2 + \frac{5}{6^2}(2 + \mathbf{E}R) + \frac{5}{6}(1 + \mathbf{E}R)\,,$$

and hence

$$\mathbf{E}R = \frac{1}{18} + \frac{5}{18} + \frac{5}{6} + \frac{35}{36}\mathbf{E}R$$

or $\frac{1}{36}\mathbf{E}R = \frac{21}{18}$, so $\mathbf{E}R = 42$.

(c) Let $Y_i, 1 \leq i \leq 5$ be the indicator for the event that the ith card is a heart. Note that $\mathbf{P}(Y_i = 1) = 1/4$. Then the number of hearts the player receives is $\sum_{i=1}^5 Y_i$ and the expected number is $\mathbf{E}\sum_{i=1}^5 Y_i = 5 \cdot \mathbf{P}(Y_i = 1) = 1/4 = 5/4$.

Solution to Problem 3. ...

(a) The sum of the PMF of a random variable over the possible values that it can take must be equal to 1. Hence, we have

$$1 = \sum_{x=-3}^{3} p_X(x) = \frac{9}{a} + \frac{4}{a} + \frac{1}{a} + \frac{1}{a} + \frac{4}{a} + \frac{9}{a} = \frac{28}{a},$$

which implies that $a = 28$. The expected value of our random variable is given by

$$\mathbf{E}[X] = \sum_x x p_X(x) = \sum_{x=-3}^{3} x \cdot \frac{x^2}{a} = -3 \cdot \frac{9}{a} - 2 \cdot \frac{4}{a} - 1 \cdot \frac{1}{a} + 1 \cdot \frac{1}{a} + 2 \cdot \frac{4}{a} + 3 \cdot \frac{9}{a} = 0.$$

(In fact, the expected value of a random variable is always equal to 0 if its PMF is even [i.e. $p_X(x) = p_X(-x)$ for all x], and it is a simple exercise to confirm this.)

(b) The following table shows the value of Z for a given value of X and the probability of that event.

x	-3	-2	-1	0	1	2	3
$p_X(x)$	9/28	1/7	1/28	0	1/28	1/7	9/28
$Z\|X = x$	9	4	1	0	1	4	9

We see that Z can take only three possible values with non-zero probability, namely 1,4, and 9. In addition, for each value, there corresponds two values of X. So we have, for example, $p_Z(9) = \mathbf{P}(Z = 9) = \mathbf{P}(X = -3) + \mathbf{P}(X = 3) = p_X(-3) + p_X(3)$. Hence the PMF of Z is given by

$$p_Z(z) = \begin{cases} 1/14 & \text{if } z = 1, \\ 2/7 & \text{if } z = 4, \\ 9/14 & \text{if } z = 9, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Recall that $\text{var}(X) = \mathbf{E}[(X - \mathbf{E}[X])]^2$, so $\text{var}(X) = \mathbf{E}[Z] = 1 \cdot \frac{1}{14} + 4 \cdot \frac{2}{7} + 9 \cdot \frac{9}{14} = 7$.

Solution to Problem 4. ...

(a) Let A be the event that a random flight is to Auckland, and C denote the event that the flight crashes. Then by the TPT, the probability a random flight crashes is

$$\mathbf{P}(C) = \mathbf{P}(C|A)\mathbf{P}(A) + \mathbf{P}(C|A^c)\mathbf{P}(A^c) = \frac{1}{10} \cdot \frac{1}{3} + \frac{1}{5} \cdot \frac{2}{3} = \frac{1}{6}.$$

(b) By part (a), each flight has probability $\frac{1}{6}$ of crashing, so the number of flights up to and *including* the first crash, call it X, is distributed according to a geometric random variable with parameter $\frac{1}{6}$. The expected number of flights *before* the first crash is $\mathbf{E}[X - 1] = \mathbf{E}X - 1 = 5$.

(c) Knowing that you've had 3 non-crash flights does not change the probability of future flights crashing since each crash is independent. Thus the answer is the same as (b).

(d) The total number Y of crashes in a year has the Binomial distribution with parameters $n = 1000$ and $p = 1/6$. The probability that Air Stanford has 1 or less crashes per year is thus

$$\mathbf{P}(Y = 0) + \mathbf{P}(Y = 1) = \left(\frac{5}{6}\right)^{1000} + 1000 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{999} \approx 1.3 \cdot 10^{-77}.$$

(e) The number of crashes is given by a $\text{Binomial}(1000, 1 - 0.9999)$ random variable, which is well-approximated by $Z \sim \text{Poisson}(\lambda)$, where $\lambda = 1000 * (1 - 0.9999) = 0.1$. Thus the desired probability is

$$\mathbf{P}(Z = 0) = \frac{0.1^0 \cdot e^{-0.1}}{0!} \approx 0.9.$$

Solution to Problem 5. ...

(a) Let R denote the profit resulting from building the circuit that requires ten component devices to not fail, where each device fails independently of the others with probability f. Letting I denote the income and C denote the cost, we can express this profit as $R = I - C$. For each option, that is the option to use either ordinary or ultra-reliable devices, C is constant but I is a random variable that has nonzero value k only if all ten component devices have not failed, which occurs with probability $(1-f)^{10}$. Thus,

$$\mathbf{E}[R] = \mathbf{E}[I - C] = \mathbf{E}[I] - C = k(1-f)^{10} - C \quad .$$

$$\begin{array}{rll} \textbf{Ordinary Option:} & C = 10, f = q = 0.10 & \Rightarrow \mathbf{E}[R] = k(0.90)^{10} - 10 \\ \textbf{Ultra-Reliable Option:} & C = 30, f = \frac{q}{2} = 0.05 & \Rightarrow \mathbf{E}[R] = k(0.95)^{10} - 30 \end{array}$$

We are interested in picking the option that gives maximum expected profit. Comparing the two expected profits we see that for $k \geq 80$ the cicuit with ultra-reliable components gives a better expected value and for $k < 80$ the circuit with ordinary components gives a better expected value.

(b) Now the probability of a circuit *not* failing as a function of a (# of ordinary devices) is given by $(0.9)^a(0.95)^{10-a}$. The cost is given by $a + 3(10-a) = 30 - 2a$. Thus

$$\mathbf{E}[R] = k(0.9)^a(0.95)^{10-a} - (30 - 2a) = k\left(\frac{0.9}{0.95}\right)^a (0.95)^{10} - 30 + 2a\,.$$

We note that this is convex in a (positive 2nd derivative) so takes its maximum at the boundary of $[0, 10]$, the support of a. This means we would always want to use one type of device in the circuit. More explicitly, we can take the derivative with respect to a of the expectd profit to get

$$\frac{d}{da}\mathbf{E}R = 2 - [k(0.95)^{10}] \cdot \alpha^a \log(1/\alpha)\,,$$

where $\alpha := 0.9/0.95$. Since increasing a decreases the magnitude of the negative term, the derivative only increases in a. So either $\mathbf{E}R$ is initially decreasing and then increasing, or it is always increasing, and in either case we would want to choose a to be an endpoint.

Solution to Problem 6. ...

(a) We compute the pmf of Z. For any $z \geq 0$ we have

$$p_Z(z) = \mathbf{P}(Z = z) = \mathbf{P}(X + Y = z) = \sum_{i=0}^{\infty} \mathbf{P}(X + Y = z | Y = i)\mathbf{P}(Y = i)$$

$$\stackrel{(a)}{=} \sum_{i=0}^{\infty} \mathbf{P}(X = z - i)\mathbf{P}(Y = i)$$

$$= \sum_{i=0}^{z} \left(\frac{\lambda_1^{z-i} e^{-\lambda_1}}{(z-i)!}\right)\left(\frac{\lambda_2^{i} e^{-\lambda_2}}{i!}\right)$$

$$= \frac{1}{z!} e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{z} \frac{z! \lambda_1^{z-i} \lambda_2^{i}}{(z-i)! i!}$$

$$= \frac{1}{z!} e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{z} \binom{z}{i} \lambda_1^{z-i} \lambda_2^{i}$$

$$\stackrel{(b)}{=} \frac{(\lambda_1 + \lambda_2)^z e^{-(\lambda_1+\lambda_2)}}{z!}$$

Here (a) follows by indepdence of X and Y and (b) follows from the binomial theorem. This shows that $Z \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

(b) We compute the joint distribution of Y, Z:

$$p_{YZ}(y, z) = \sum_{i=0}^{\infty} \mathbf{P}(Y = y, Z = z | X = i)\mathbf{P}(X = i)$$

$$= \binom{y+z}{y} p^y (1-p)^z \frac{\mu^{y+z} e^{-\mu}}{(y+z)!}$$

where only the term $i = y + z$ in the sum was retained because the other terms evaluate to zero. Continuing, rearranging the above gives

$$= \frac{(y+z)!}{y! z!} (\mu p)^y ((1-p)\mu)^z e^{-\mu p} e^{-\mu(1-p)} \frac{1}{(y+z)!}$$

$$= \left(\frac{(\mu p)^y e^{-\mu p}}{y!}\right)\left(\frac{((1-p)\mu)^z e^{-\mu(1-p)}}{z!}\right).$$

But this is just the product of two Poisson pmfs, so summing over z removes the 2nd factor and shows that $Y \sim \text{Poisson}(\mu p)$ and similarly for Z. Also Y, Z are independent because their joint pmf factorizes.

Solution to Problem 7. ...

(a) We know the inequality holds for f convex and $n = 2$. Suppose the inequality holds for a convex combination of $n-1$ reals. Then we have for $n \geq 3$

$$\begin{aligned} f\left(\sum_{i=1}^{n} \lambda_i x_i\right) &= f\left((\lambda_1+\lambda_2)\left[\frac{\lambda_1}{\lambda_1+\lambda_2}x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}x_2\right] + \sum_{i=3}^{n} \lambda_i x_i\right) \\ &\leq (\lambda_1+\lambda_2) f\left(\frac{\lambda_1}{\lambda_1+\lambda_2}x_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}x_2\right) \\ &\quad + \sum_{i=3}^{n} \lambda_i f(x_i) \\ &\leq \sum_{i=1}^{n} \lambda_i f(x_i)\,, \end{aligned}$$

where the last step just applies the result $n = 2$.

(b) If X is discrete with uncountable support then we need to be careful because the inequality we proved in part (a) does not deal with this case. You get full credit on the problem even if you skipped over this issue, but a brief sketch of a way around this is included here. We need to assume that $\mathbf{E}X < \infty$, i.e. $\mathbf{E}X$ is finite. Then, given any $\epsilon > 0$, we can choose a finite subset $A \subset \mathcal{X}$ of the support of X so that $|\mathbf{E}[X|X \in A] - \mathbf{E}(X)| < \epsilon$. We can prove the inequality with an error ϵ and the expectation conditioned on A. Taking an appropriate sequence of sets A so that $\epsilon \to 0$ solves the issue. But we assume that X has finite support. By part (a) we have that

$$\begin{aligned} f(\mathbf{E}X) &= f\left(\sum_{x\in\mathcal{X}} x p_X(x)\right) \\ &\leq \sum_{x\in\mathcal{X}} p_X(x) f(x)\,, \end{aligned}$$

but this last sum is just $\mathbf{E}f(X)$ so we're done.

(c) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote the function $f(x) = 1/x$. Then by independence of X and Y we have $\mathbf{E}[X/Y] = (\mathbf{E}X)(\mathbf{E}f(Y))$. Now f is convex (take the 2nd derivative), so Jensen's inequality (part (b)) gives that $\mathbf{E}f(Y) \geq f(\mathbf{E}Y) = 1/\mathbf{E}Y = 1/\mathbf{E}X$, where the last equality is because X and Y have the same distribution. Combined with the earlier line this shows the desired inequality.