UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

**Problem Set 3**
Spring 2011

**Issued:** Tuesday, February 1, 2011 **Due:** In class Tuesday, Feb 8, 2011

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**Reading:** Bertsekas & Tsitsiklis, §1.5, §1.6

We will use the following notation:

$$[n] := \{1, 2, \ldots, n\}.$$

*Problem* 1. Each of the following should be a straight-forward counting problem.

(a) The weather on any given day can be sunny, cloudy, rainy, or snowy. Assume four sequential seasons (fall, winter, spring, summer), that a snowy day can happen only during the winter, that a rainy day cannot happen in the summer, and that each season has 90 days. What is the number of all possible distinct 360-day weather sequences (consecutive days)?

(b) Mary and Tom park their cars in an empty parking lot that consists of $N$ parking spaces in a row, where $N \geq 4$. Assume that each possible pair of parking locations is equally likely. Calculate the probability that the parking spaces they select are exactly 4 apart (that is, exactly three empty spaces between them).

(c) An urn contains 40 balls, 10 of which are red. Suppose that 10 balls are selected at random, without replacement. What is the probability that exactly 5 of the selected balls are red?

(d) The word "drawer" is spelled with six scrabble tiles. The tiles are then randomly rearranged. What is the probability of the rearranged tiles spelling the word "reward"?

*Problem* 2. Determine whether each of the following statements is true. Argue if true, give a counterexample or brief explanation if false. The symbols $A$ and $B$ stand for events. Don't forget to consider the null event, which has probability zero.

(a) An event $A$ can not be independent of itself.

(b) Two disjoint events are always independent.

(c) Suppose that $A$ is a subset of $B$. Then, $A$ and $B$ can be independent.

(d) $A$ and $B$ are independent if and only if $A$ and $B^c$ are.

(e) The events $\emptyset$ (empty set) and $\Omega$ (whole sample space) are independent of any other event $A \subset \Omega$, and these are the only two events with this property.

(f) $A$ and $B$ are independent if $P(B|A) = P(B|A^c)$.

*Problem* 3. $k$ numbers are chosen at random, one after the other, without replacement, from the set $[n] = \{1, \ldots, n\}$. What is the probability that

(a) the set of numbers chosen forms a group of consecutive numbers?

(b) the numbers are chosen in ascending order and form a group of consecutive numbers?

(c) the number are chosen in ascending order?

*Problem* 4. Let $A$ and $B$ be independent events with $0 < kP(B) = P(A) < 1$, where $k \geq 1$. Let $C = (A \cap B^c) \cup (A^c \cap B)$ be the event that exactly one of $A$ and $B$ occurred. Show that

$$P(A \mid C) \geq \frac{k}{k+1}.$$

*Problem* 5. Recall what it means for two events to be independent. We now extend this to collections (or classes) of events. Consider two collections of events $\mathcal{A} = \{A_1, A_2, \ldots\}$ and $\mathcal{B} = \{B_1, B_2, \ldots\}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are independent if for any $A_i \in \mathcal{A}$ and any $B_j \in \mathcal{B}$, $A_i$ and $B_j$ are independent.

(a) Someone claims that events $A$ and $B$ are independent if and only if $\{A, A^c, \emptyset, \Omega\}$ and $\{B, B^c, \emptyset, \Omega\}$ are independent as collections. Is this true?.

(b) Suppose that $\{A, B, A \cap B\}$ and $\{C\}$ are independent as collections. Show that $A \cup B$ and $C$ are independent events. (You might find it helpful to answer the next part first.)

(c) Under the assumptions of part (b), what can you say about the independence of $A \cap B^c$ and $C$?

(d) In general, what is the maximum number of "distinct" events that you can get from $A$ and $B$ by taking unions, intersections and complements? (Any number of such operations are allowed in any order.) Under the assumptions of part (b), how many of these do you think are independent of $C$? (A guess for this later part is enough.)

(e) Suppose in part (d) that you know in addition that $A \subset B$. How does this affects your answers?

(f) Suppose in part (b), you in addition know that $A$ and $B$ are independent. Is it true then that $A$, $B$ and $C$ are (jointly) independent?

*Problem* 6. We pick a point at random form the triangle in the plane ($\mathbb{R}^2$) with vertices $(0,0)$, $(0,2)$ and $(2,0)$. Let us denote a point in the plane by $p$ and its Cartesian coordinates by $x(p)$ and $y(p)$, respectively. Thus, the sample space is

$$\Omega := \{p \in \mathbb{R}^2 \;\mid\; x(p) \geq 0,\;\; y(p) \geq 0,\;\; x(p) + y(p) \leq 2\big\}.$$

Our probability model is as follows: assign to a subset $A \subset \Omega$, a probability proportional to its area, that is,

$$P(A) = \frac{\text{area}(A)}{\text{area}(\Omega)} = \frac{1}{2}\text{area}(A).$$

For any $\alpha, \beta \in [0,2]$, define the events

$$X_\alpha := \big\{p \in \Omega \;:\; x(p) \leq \alpha\big\}, \quad Y_\beta := \big\{p \in \Omega \;:\; y(p) \leq \beta\big\}.$$

(a) Are the events $X_{\frac{1}{2}}$ and $Y_{\frac{1}{2}}$ independent?

(b) Conditioned on $X_1 \cap Y_1$, are the two events of part (a) independent?

*Problem* 7. Imagine that 8 rooks are being placed on an $8 \times 8$ chessboard. The rooks are in *non-attacking position* if there is no row or column with more than one rook. In how many ways the rooks can be placed on the board? How many of these are non-attacking? Answer in each of the following cases:

(a) the rooks are indistinguishable,

(b) the rooks are distinguishable (say, labeled $1, \dots, 8$).

Now, suppose that the rooks are placed randomly on the board (say, one at time). What it the probability that they end up in a non-attacking position?

*Problem* 8. A multiset is an object like a set in which the elements are allowed to be repeated. Consider the multiset $M_n := \{1,1,2,2,\dots,n,n\}$ which contains each of the elements $1,2,\dots,n$ twice. Consider permutations of (elements of) $M_n$. For example, all the permutations if $M_2$ are 1122, 2211, 1212, 2121, 1221, 2112.

(a) What is the number of permutations of $M_n$?

(b) What is the number of permutations of $M_n$ with consecutive 1's?

(c) What is the number of permutations of $M_n$ in which the two letters in $M_n$ corresponding to each of the elements of $T := \{1,2,\dots,i\}$ appear consecutively? (That is, 1's appear consecutively, 2's appear consecutively and so on.) Argue that your answer only depends on the size of $T$ not the particular elements chosen.

*Problem* 9. Suppose that $2n$ customers stand in line at a box office, $n$ with 5-dollar bills and $n$ with 10 dollar bills. Suppose each ticket costs 5 dollars, and the box office has no money initially. What is the probability that box office always has enough change to give back to \$10 customers?

> *Hint:* You may find ideas of this week's discussion sections to be helpful (e.g., Ballot Theorem and Reflection Principle)