UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 3 Solutions
Spring 2011

Solution to Problem 1. ...

(a) In spring and autumn, each day has one of three possible types of weather. Therefore, for each of these two seasons the total number of possible weather sequences is 3^{90}. Similarly, there are 4^{90} possible sequences for winter and 2^{90} possible sequences for summer. The total number of possible sequences is:

$$(3)^{90}(3)^{90}(4)^{90}(2)^{90} \approx 10^{167}.$$

(b) Mary may choose any of the N parking spaces. For each of these spaces, Tom may park in any of the remaining $N-1$ spaces. So the total number of outcomes is $N(N-1)$.

There are $N-4$ possible parking configurations where Mary parks to the left of Tom and there are exactly three spaces between them. Similarly, there are $N-4$ possible parking configurations where Tom parks to the left of Mary and there are three spaces between them. So the number of outcomes in A is $2(N-4)$. Therefore, using the discrete uniform law,

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{2(N-4)}{N(N-1)}.$$

(c) First, we evaluate how many ways there are to select 10 balls from an urn of 40 balls without replacement. There are $\binom{40}{10}$ ways to accomplish this. Next we evaluate how many of the above mentioned combinations contain 5 red balls and 5 that are not red. There are $\binom{10}{5}$ ways to select 5 red balls from 10, and for each one of these combinations, there are $\binom{30}{5}$ ways to select the remaining 5 balls. Therefore, we conclude that $\binom{10}{5}\binom{30}{5}$ out of a possible $\binom{40}{10}$ combinations contain 5 red balls and 5 that are not red. Thus the

$$P(\text{exactly 5 red balls}) = \frac{\binom{10}{5}\binom{30}{5}}{\binom{40}{10}}.$$

(d) Suppose that we mark the tiles 1 to 6, from left to right; hence, "d" is tile 1, the first "r" is tile 2 and so on. The number of all possible rearrangements are the number of permutations of [6], which is 6!. The word "reward" corresponds to the two permutations 254361 and 654321. (We have two options for the first "r", namely tiles 2 and 6, and exactly 1 option for the rest of the letters.) Hence the desired probability is $\frac{2}{6!} = \frac{1}{360}$.

Solution to Problem 2. An event which has zero probability is call a *null* event. For this problem what you need to know is that there could be plenty of null events on a probability space.

(a) FALSE. The condition for independence of A of itself is $P(A \cap A) = P(A)P(A)$, which gives $P(A)\big(1 - P(A)\big) = 0$. This holds iff $P(A) = 0$ or $P(A) = 1$. Hence, an event A is independent of itself iff A or A^c is a null event.

(b) FALSE. For two events A and B with $A \cap B = \emptyset$, the independence condition is $P(\emptyset) = P(A)P(B)$. Since $P(\emptyset) = 0$, this holds iff either A or B is a null event.

(c) TRUE. Under the assumption $A \subseteq B$, we have $A \cap B = A$. The independence criterion reads $P(A)\big(1 - P(B)\big) = 0$. This holds iff A or B is null event. (Note that if B is null, then A, being a subset of B, is also null in this case.)

(d) TRUE. That independence of A and B implies that of A and B^c, as was discussed in the lecture. (Just note that $P(A \cap B^c) = P(A) - P(A \cap B) = P(A)\big(1 - P(B)\big) = P(A)P(B^c)$.) Using this same fact and applying it to A and $(B^c)^c = B$ we get the reverse statement.

(e) FALSE. The statement about $\emptyset$ and Ω is true, but these are not necessarily the only sets with this property. Suppose that an event B is independent of any other event A. In particular, it is independent of itself. By part (a), either A or A^c should be null. It is not hard to verify that this is also sufficient.

(f) TRUE. Assume the condition holds. Let $\alpha := P(B|A) = P(B|A^c)$. By the total probability theorem,

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = \alpha P(A) + \alpha P(A^c) = \alpha.$$

That is, $P(B) = P(B|A)$ which implies independence.

Solution to Problem 3. ...

(a) The probability of choosing any given group of numbers is $\binom{n}{k}^{-1}$. Of these $n - k + 1$ are groups of consecutive numbers. Hence, the answer is $\frac{n-k+1}{\binom{n}{k}}$.

(b) The first number chosen must be one of $\{1, \ldots, n-k+1\}$, and the probability of this is $\frac{n-k+1}{n}$. Each of the remaining $k-1$ numbers is now uniquely specified and the probability of choosing precisely these numbers is $\frac{1}{n-1}\frac{1}{n-2}\cdots\frac{1}{n-k+1}$. Hence the answer is $\frac{n-k+1}{k!\binom{n}{k}}$.

(c) For any group of k numbers chosen, the probability that these numbers were chosen in ascending order is $(k!)^{-1}$. Hence the answer is $\frac{1}{k!}$.

Note that the events $\{$ A group of consecutive numbers is chosen $\}$ and $\{$ The number are chosen in ascending order $\}$ are independent events.

Solution to Problem 4. Recall that independence of A and B implies independence of A and B^c as well as that of A^c and B. To simplify notation, let $\alpha := P(A)$ and $\beta := P(B)$, so that $\alpha = k\beta$. Then $k \geq 1$, implies $\alpha \geq \beta$. We can write

$$P(A \cap C) = P(A \cap B^c) = P(A)P(B^c) = \alpha(1-\beta),$$

and

$$P(A^c \cap B) = P(A^c)P(B) = (1-\alpha)\beta \leq (1-\beta)\beta.$$

Hence, we have

$$P(C) = P(A \cap B^c) + P(A^c \cap B) \leq (\alpha+\beta)(1-\beta).$$

Combining the pieces, we get

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} \geq \frac{\alpha(1-\beta)}{(\alpha+\beta)(1-\beta)} = \frac{\alpha}{\alpha+\beta} = \frac{k}{k+1}.$$

Solution to Problem 5. ...

(a) The claim is true. As was shown in lecture, A being independent of B implies independence of each of the pairs A, B^c and A^c, B and A^c, B^c. Furthermore, $\emptyset$ and Ω are independent of every event.

(b) Let $A_1 := A \setminus B := A \cap B^c$, $B_1 := B \setminus A := B \cap A^c$ and $D := A \cap B$. Note that A is the disjoint union of A_1 and D. Hence,

$$\begin{aligned} P(A)P(C) = P(A \cap C) = P\big((A_1 \cup D) \cap C\big) &= P\big(A_1 \cap C) + P\big(D \cap C\big) \\ &= P\big(A_1 \cap C) + P(D)P(C) \end{aligned}$$

where the first and the last equalities follows from our independence assumption. Note that $P(A_1) = P(A) - P(D)$. Rearranging the above gives

$$P(A_1)P(C) = P(A_1 \cap C),$$

showing that A_1 and C are independent. By symmetry (or if you like repeating the same argument with A replaced everywhere with B), we get that B_1 and C are also independent. Now we have

$$\begin{aligned} P\big((A \cup B) \cap C\big) = P\big((A_1 \cup D \cup B_1) \cap C\big) &= P(A_1 \cap C) + P(D \cap C) + P(B_1 \cap C) \\ &= P(A_1)P(C) + P(D)P(C) + P(B_1)P(C) \\ &= \big\{\underbrace{P(A_1) + P(D) + P(B_1)}_{P(A \cup B)}\big\}P(C) \end{aligned}$$

which is the desired result.

(c) They are independent as part (b) shows.

(d) Let A_1, B_1 and D be as defined in part (b) and let $E := A^c \cap B^c = \Omega \setminus (A \cup B)$. Note that these four events A_1, B_1, D and E partition the sample space. In general, all four are non-empty (corresponding to the maximum number of distinct generated subsets). Any set generated by the set-theoretic operations out of A and B has a simple relation with respect to any of these four events: it either contains it entirely or excludes it entirely. Hence there are $2^4 = 16$ distinct subsets possible. As might be guessed form the argument of part (b), all of them are independent of C.

(e) Assumption $A \subset B$ implies $A_1 = \emptyset$. Hence, we are effectively dealing with a partition of Ω by 3 sets B_1, D and E. Hence, the number of generated subsets is reduced to $2^3 = 8$.

(f) The claim is true. Part (b) guarantees pairwise independences of A, C and B, C. This part guarantees the other necessary pairwise independence, i.e., A, B. Finally, we have $P(A \cap B \cap C) = P(A \cap B)P(C) = P(A)P(B)P(C)$, where the first inequality follows by independence of $A \cap B$ and C. Hence all the criteria for independence of three events are met.

Solution to Problem 6. ...

(a) The set $X_{\frac{1}{2}}$ corresponds to a trapezoid in the plane with the area $\frac{1}{2}(\frac{3}{2}+2)\frac{1}{2} = \frac{7}{8}$. Hence $P(X_{\frac{1}{2}}) = \frac{7}{16}$. By symmetry, $P(Y_{\frac{1}{2}}) = \frac{7}{16}$. The event $X_{\frac{1}{2}} \cap Y_{\frac{1}{2}}$ corresponds to a square in $\mathbb{R}^2$, with area $(\frac{1}{2})^2 = \frac{1}{4}$. Hence $P(X_{\frac{1}{2}} \cap Y_{\frac{1}{2}}) = \frac{1}{8}$. It is now clear that the two events $X_{\frac{1}{2}}$ and $Y_{\frac{1}{2}}$ are not independent, since

$$P(X_{\frac{1}{2}} \cap Y_{\frac{1}{2}}) = \frac{1}{8} \neq \frac{49}{256} = P(X_{\frac{1}{2}})P(Y_{\frac{1}{2}}).$$

(b) The event $X_1 \cap Y_1$ is a square of side length 1 in $\mathbb{R}^2$, which has an area of 1. Conditioning on this event effectively restricts our sample space to this square. Taking $\Omega' := X_1 \cap Y_1$ as our new sample space and $P'(\cdot) := P(\cdot \mid \Omega')$ as our new probability law, we conclude that $P'(A) = \text{area}(A)$, for $A \subset \Omega'$. It follows that $P'(X_{\frac{1}{2}}) = P'(Y_{\frac{1}{2}}) = \frac{1}{2}$ and $P'(X_{\frac{1}{2}} \cap Y_{\frac{1}{2}}) = \frac{1}{4}$. Hence, conditional on $X_1 \cap Y_1$, the two events $X_{\frac{1}{2}}$ and $Y_{\frac{1}{2}}$ are independent, since

$$P'(X_{\frac{1}{2}} \cap Y_{\frac{1}{2}}) = \frac{1}{4} = P'(X_{\frac{1}{2}})P'(Y_{\frac{1}{2}}).$$

Solution to Problem 7. ...
We will use the *falling factorial* notation $(x)_n$ in this problem:

$$(x)_n := x(x-1)(x-2)\cdot(x-n+1) = \frac{x!}{(x-n)!} = \frac{1}{n!}\binom{x}{n}.$$

(a) For the case where the rooks are indistinguishable, we can count the non-attacking positions as follows: for the first row of the board, there a 8 possible states, corresponding to which column is being occupied by a rook. Having placed a rook in the first row, there are 7 possibilities for the second row, as the first rook rules out one of the columns. Continuing this way, we arrive at the number

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 8!$$

for the non-attacking positions. (Note that we are effectively counting 8-by-8 matrices having exactly one 1 in each row and each column and 0's elsewhere. These matrices are called *permutation matrices* and as is roughly suggested by the above, there is a bijection between them an permutations of [8].)

To count the total number of (unrestricted) configurations, we note that each configuration corresponds to selecting 8 of the 64 possible positions on the board for being occupied by the rooks. Hence, we have

$$\binom{64}{8}$$

for the total number of configurations.

(b) For the case where the rooks are distinguishable, we can count the non-attacking positions as follows: for the rook labeled 1 there are $8 \times 8 = 64$ possible positions. Once rook 1 is placed, there are $7 \times 7 = 49$ possible positions to place rook 2, as rook 1 rules out the row and column it sits on. Continuing in this manner, we arrive at the number

$$8^2 \cdot 7^2 \cdot 6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2 = \prod_{i=1}^{8} i^2$$

for the number of non-attacking positions. Another way to count is as follows: for each of the configurations of part (a), there are 8! configurations of part (b), corresponding to the permutations of the 8 rooks among themselves without changing the occupancy pattern on the board. Hence, the number of non-attacking configurations for part (b) is

$$8! \cdot 8! = \prod_{i=1}^{8} i^2.$$

The total number of (unrestricted) configurations is

$$64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57 = (64)_8$$

using the falling factorial notation.

Now, randomly placing rooks on the board, one at a time, closely relates to the way of counting we employed in part (b). We get a (discrete) uniform distribution on the $(60)_8$ possible configurations, out of which $\prod_{i=1}^{8} i^2$ of them are favorable to use. Hence the desired probability is

$$\frac{\prod_{i=1}^{8} i^2}{(60)_8}.$$

Solution to Problem 8. (a) It should be clear from the discussion in the lecture, that the number of permutations of M_n is

$$\binom{2n}{2,2,\dots,2} = \frac{(2n)!}{\prod_{i=1}^{n} 2!} = (2n)!\,2^{-n}.$$

(For example, an interpretation could be that we are placing $2n$ possible positions in a word of length $2n$ into n groups of size 2 each. Or, we have $(2n)!$ permutations disregarding repetitions and then we have to divide by $\prod_{i=1}^{n} 2!$, since permutations of like letters lead to the same word.)

(b) Gluing the two 1's together, the desired number is that of permutations of the multiset $\{1,2,2,3,3,\dots,n,n\}$ which is equal to

$$\binom{2n-1}{1,2,\dots,2} = (2n-1)!\,2^{-(n-1)}$$

(c) By an argument similar to the previous part, gluing the numbers that have to appear next to each other, we have that the desired number is that of permutations of $\{1,2,\dots,i,i+1,i+1,\dots,n,n\}$, that is,

$$\binom{2n-i}{\underbrace{1,1,\dots,1}_{i\text{ times}},\underbrace{2,2,\dots,2}_{n-i\text{ times}}} = (2n-i)!\,2^{-(n-i)}.$$

By symmetry, the number above only depends on the size of the set $\{1,\dots,i\}$ of letters glued together.

Solution to Problem 9. ...
By $\mathbb{Z}$ we mean the set of integers, i.e., $\mathbb{Z} := \{\dots,-2,,-1,0,+1,+2,\dots\}$. Recall that $[2n] := \{1,2,\dots,2n\}$.

Direct approach using reflection principle. For $i \in [2n]$, let

$$X_i := \begin{cases} +1 & \text{if customer } i \text{ has a \$5 bill} \\ -1 & \text{if customer } i \text{ has a \$10 bill} \end{cases}$$

and $S_k := \sum_{i=1}^{k} X_i$. Define $S_0 := 0$. Note that we also have $S_{2n} = 0$, as the number of \$5 customers and \$10 customers are the same.

It is not hard to see that box office always has enough change for \$10 customers iff

$$S_k \geq 0, \quad \text{for all } k \in [2n]. \tag{1}$$

Consider a path in $\mathbb{Z}^2$, joining the points $\{(k,S_k) : 0 \leq k \leq 2n\}$. Note that all such paths start at $(0,0)$ and end at $(2n,0)$. Let $\mathcal{P}$ be the collection of all such paths. Condition (1) is equivalent to the path always staying above the x-axis or touching it (but never going below it). Let $\mathcal{G}$ stand for the collection of all these "good" paths, that is, those paths in $\mathcal{P}$ satisfying (1).

To count the total number of paths, $|\mathcal{P}|$, we note that any such path is determined by the corresponding vector of ± 1's, i.e., $(X_1, X_2, \dots, X_{2n})$, and any such vector is

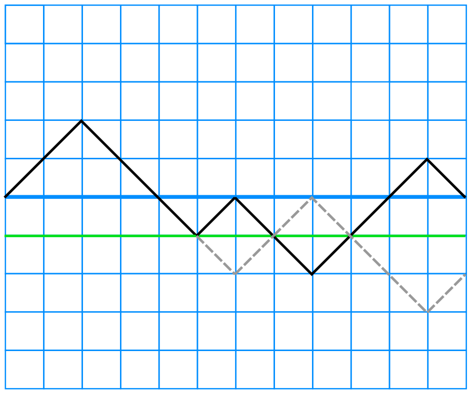

Figure 1: Reflecting a portion of a "bad" path

determined by the positions of, say, +1's. There are $\binom{2n}{n}$ ways of choosing the n positions for +1's among the $2n$ possible positions, hence

$$|\mathcal{P}| = \binom{2n}{n}.$$

To count the set of good paths, we count its complement, the set of "bad" paths, i.e., those crossing the x-axis. To count these, we establish a bijection between the set of all bad paths in $\mathcal{P}$, and all the (unrestricted) paths that start at $(0,0)$ and end at $(2n,-2)$. This is done using the reflection trick, similar to the one used to prove the Ballot theorem in the discussion of this week.

Consider a bad path starting at $(0,0)$, ending at $(2n,0)$ and having $X_m = -1$ for some $m \in [2n-1]$. Thus the path hits the x-axis at $x = m-1$. Reflecting the portion of the path to right of $x = m$ across the horizontal line $\{(x,y) : y = -1\}$, we obtain a path which ends at $(2n,-2)$ (and of course still starts at $(0,0)$). Going in the reverse direction, starting with any path that starts at $(0,0)$ and ends at $(2n,-2)$, there is a smallest x_0 at which the path crosses x-axis. Hence the path is at $(x_0+1,-1)$ in the next step. Reflecting the portion of the path to the right of x_0+1 across the horizontal line mentioned before gives us a bad path with endpoints $(0,0)$ and $(2n,0)$ (cf. Fig. 1). It should also be clear the two operations are indeed the inverse of each other. Hence, we have established our bijection.

Now, the number of paths with endpoints $(0,0)$ and $(2n,-2)$ is the number of vectors $(Y_1, Y_2, \ldots, Y_{2n})$ with $n-1$ plus-ones and $n+1$ minus-ones (as they should sum to -2). Hence, the number is $\binom{2n}{n-1}$. This is also the number of bad paths in $\mathcal{P}$. Hence, the number of good paths is

$$|\mathcal{G}| = \binom{2n}{n} - \binom{2n}{n-1} = \binom{2n}{n}\left(1 - \frac{n}{n+1}\right) = \frac{1}{n+1}\binom{2n}{n} =: C_n.$$

The numbers C_n are the famous Catalan numbers. The good paths in our problems are sometimes called Dyck paths. Thus, we have shown that Catalan numbers count the number of Dyck paths.

The desired probability is obtained by dividing the number of favorable outcomes to the total number of outcomes (as we are dealing with a uniform probability law), hence,

$$\frac{|\mathcal{G}|}{|\mathcal{P}|} = \frac{1}{n+1}$$

is the answer.

Alternative approach using the Ballot Theorem. Recall the Ballot Theorem from the discussion of this week. There, the corresponding paths were restricted to stay strictly above the x-axis. The Dyck paths considered in this problem are allowed to also touch the x-axis. By adding an extra $+1$ (or upward move) to the beginning of a Dyck path we obtain a Ballot path with parameters $a = n + 1$ and $b = n$. Conversely, removing an upward move from the start of such Ballot path we get a Dyck path. (Recall that a Ballot path should start with an upward move.) We showed the number of Ballot paths with parameter a and b to be

$$\frac{a-b}{a+b}\binom{a+b}{a}.$$

Substituting $a - 1 = b = n$, we get

$$\frac{1}{2n+1}\binom{2n+1}{n+1} = \frac{1}{2n+1}\frac{(2n+1)!}{(n+1)!n!} = \frac{1}{n+1}\frac{(2n)!}{n!n!} = C_n$$

for the number of Dyck paths.