EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2011
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Final Exam

Last name	First name	SID

***Rules*.**

- You have 180 mins (7:05pm - 10:05pm) to complete this exam.
- The exam is not open book, but you are allowed one sheet of handwritten notes; calculators will be allowed.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

***Please read the following remarks carefully*.**

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.
- The maximum you can score is 105.
- Chocolate Chip cookies will be provided

Good Luck!

Problem	Points earned	out of
Problem 1		14
Problem 2		8
Problem 3		24
Problem 4		20
Problem 5		20
Problem 6		14
Extra Credit		5
Total		105

Problem 1 [14] Multiple Choice. No partial credit will be given for this problem.

(a) [2] Bob rolls two 12 sided dice. Let the outcomes be A and B. The following events are independent:

(a) $A = 4$ and $A + B \geq 13$
(b) $A = 5$ and $A + B < 13$
(c) $A = 9$ and $A + B = 13$
(d) $A = 7$ and $A + B = 16$
(e) They can never be independent

(b) [3] A Cookie Factory wants to ensure that the number of chocolate chips in a cookie behaves like a Poisson Random Variable, X, distributed with mean 10. The chocolate chips are mixed into the cookie batter and the cookies are then baked.

(i) [1] The average of the square of the number of chocolate chips found in a randomly selected cookie should be
(a) 110
(b) 100
(c) 121

(ii) [2] A tiny sensor is attached to a random chocolate chip. The expected number of chips found on the cookie that has the sensor should be
(a) 10
(b) 11
(c) 11.5
(d) 9.5

(c) [2] For each of the following sequences, write "True" if it converges in probability and write "False" otherwise.

(a) $\{X_n\}$ where each X_n is distributed as $N(0, 1/n)$
(b) $\{M_n\}$ where each $M_n = (Y_1{+}Y_2{+}...{+}Y_n)/n$ with each Y_i distributed as a Bernoulli(p)
(c) $D_n = (U_1 + ... + U_n)/(U_1 + ... + U_{2n})$ where each U_i is uniformly distributed in $(0, 1)$
(d) $W_n = |Z|^n$ where Z is distributed as N(0,1)

(d) [4] X is a uniform random variable on $[\mu - a, \mu + a]$, i.e.

$$f_X(x) = \begin{cases} \frac{1}{2a}, & \mu - a \leq x \leq \mu + a \\ 0 & \text{otherwise}. \end{cases}$$

You observe n i.i.d. samples, $X_1, X_2, \ldots, X_n$ of X.

(i) [2] What is the maximum likelihood estimate for μ?
(a) $\hat{\mu}_{\text{ML}} = \frac{X_1+X_2+\cdots+X_n}{n}$
(b) $\hat{\mu}_{\text{ML}} = \frac{X_1+X_n}{2}$
(c) $\hat{\mu}_{\text{ML}} = \frac{\max\{X_1,...,X_n\}+\min\{X_1,...,X_n\}}{2}$
(d) $\hat{\mu}_{\text{ML}} = \frac{\max\{X_1,...,X_n\}+\min\{X_1,...,X_n\}}{n}$

(ii) [2] What is the maximum likelihood estimate for a?

(a) $\hat{a}_{\text{ML}} = \frac{\max\{X_1,\ldots,X_n\} - \min\{X_1,\ldots,X_n\}}{2}$
(b) $\hat{a}_{\text{ML}} = \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{n}$
(c) $\hat{a}_{\text{ML}} = \frac{X_1 + X_2 + \cdots + X_n}{n}$
(d) $\hat{a}_{\text{ML}} = \frac{(X_1 - X_2)^2 + \cdots + (X_{n-1} - X_n)^2}{n}$

(e) [3] A coin is tossed repeatedly (independent trials) until a Head is observed for the first time. X denotes the number of trials needed to observe the first Head. The two hypotheses are

- $H_1 : X \sim \text{Geometric}(p_1)$
- $H_0 : X \sim \text{Geometric}(p_0)$

where p_1 and p_0 are known numbers with $p_1 < p_0$. Let π_0 and $\pi_1 = 1 - \pi_0$ respectively denote the a priori probabilities of hypotheses H_0 and H_1 and assume that $0 < \pi_0 < 1$.

(i) [1] We observe $X = 1$. For what value of π_0, does the MAP rule choose hypothesis H_1?

(a) $\pi_0 > \frac{p_0 + p_1}{p_0 p_1}$
(b) $\pi_0 < \left(\frac{p_0}{p_1} + 1\right)^{-1}$
(c) $\pi_0 > \frac{p_1}{p_0}$
(d) No value of π_0

(ii) [1] For what values of π_0 (if any) does the MAP decision rule always choose hypothesis H_1 regardless of the value of the observation X?

(a) $\pi_0 > \frac{p_0 + p_1}{p_0 p_1}$
(b) $\pi_0 < \left(\frac{p_0}{p_1} + 1\right)^{-1}$
(c) $\pi_0 > \frac{p_1}{p_0}$
(d) No value of π_0

(iii) [1] For what values of π_0 (if any) does the MAP decision rule always choose hypothesis H_0 regardless of the value of the observation X?

(a) $\pi_0 > \frac{p_0 + p_1}{p_0 p_1}$
(b) $\pi_0 < \frac{1}{2}\left(\frac{p_0}{p_1} + \frac{1}{2}\right)^{-1}$
(c) $\pi_0 > \frac{p_1}{p_0}$
(d) No value of π_0

Problem 2 [8] Markov Chains

Alice asks Bob if he would like to play the following game which consists of rounds. Each time a round is played, Bob wins 1 dollar if he wins, and loses 1 dollar if he loses.

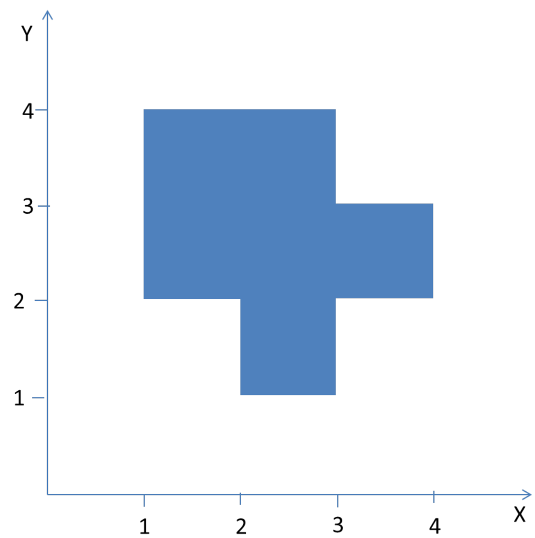
The game keeps track of Bob's winnings in the rounds he has played. If his winnings are a multiple of 3 he flips a coin which comes up heads with probability p_1; otherwise he flips a coin that comes up heads with probability p_2. Bob wins the round if the coin he has flipped comes up heads.

(a) [4] Formulate a Markov Chain with three states that allows you determine the fraction of time he flips each coin in steady state. Write down the balance equations but DO NOT solve for the steady state probabilities.

(b) [4] Write down an expression in terms of the steady state probabilities that will determine whether or not the game is a winning game for Bob (i.e. whether the prob that he will win a round in steady state is at least 0.5.

(c) [2] Extra Credit: Solve the equations in part (a) and determine if the game is a winning game for Bob when $p_1 = 0.09$ and $p_2 = 0.74$.

Problem 3 [24] Continuous Random Variables and Estimation The joint distribution of X and Y is uniform over the shaded area in the figure.



(a) [5] Draw the marginal distributions $f_X(x)$ and $f_Y(y)$.

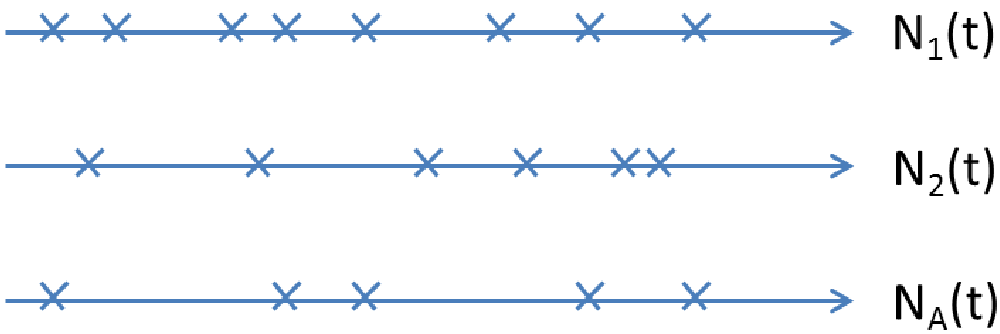
(b) [4] Suppose we can observe Y and want to use it to estimate X. Find the Minimum Squared Error Estimation of X.

(c) [5] What is $cov(X,Y)$?

(d) [5] Find the Bayes Linear Least Square estimate of X based on Y.

(e) [5] Find the CDF of Z=X+Y given that $X \in [2,3]$ and $Z \leq 6$. HINT: Draw the line $X+Y=k$ for various values of k and see what happens. Don't compute integrals!

Problem 4 [20[Let $\{N_1(t), t \geq 0\}$ be a Poisson process of rate λ. Assume that the arrivals from this process are switched on and off by a second independent Poisson Process $\{N_2(t), t \geq 0\}$ of rate γ.



Let $\{N_A(t), t \geq 0\}$ be the switched process. $N_A(t)$ includes the arrivals from $\{N_1(t), t \geq 0\}$ when $N_2(t)$ is even and excludes arrivals from $\{N_1(t), t \geq 0\}$ when $N_2(t)$ is odd.
NOTE: Please be clear about when you are using merged/split Poisson Processes.

(a) [5] What is the pmf for the number of number of arrivals of $\{N_1(t), t \geq 0\}$ in the first period when the switch is on (i.e. until the first arrival of $\{N_2(t), t \geq 0\}$.

(b) [5] Given that the first arrival for $\{N_2(t), t \geq 0\}$ is at time τ, find the conditional pmf for the number of arrivals of $\{N_1(t), t \geq 0\}$ for the first period that the switch is on.

(c) [5] Given that the number of arrivals in the first period when the switch is on is n find the density for end of that first period (for the time u when $\{N_2(t), t \geq 0\}$ has its first arrival).

(d) [5] Find the expected time between arrivals for the process $\{N_A(t), t \geq 0\}$

Problem 5 [20] Discrete Random Variables and Counting: A deck of 52 cards (with four aces) is well shuffled and placed face down on a table. Bob turns up cards one by one from the top of the deck until an ace is revealed.

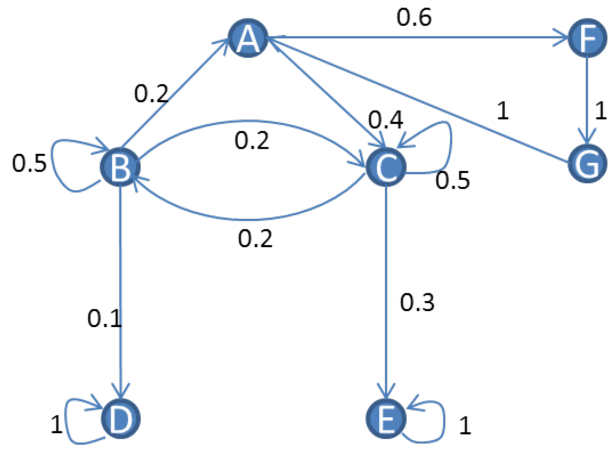
(a) [4] What is the probability that Bob will turn up a total of four cards?

(b) [4] Given that Bob has already turned up 3 cards, what is the probability that the fifth card will be an ace?

(c) [8] Show that the expected number of cards that Bob turns up is 10.6. HINT: Use the fact that the four aces divide the non-ace cards into groups. Then use indicator random variables.

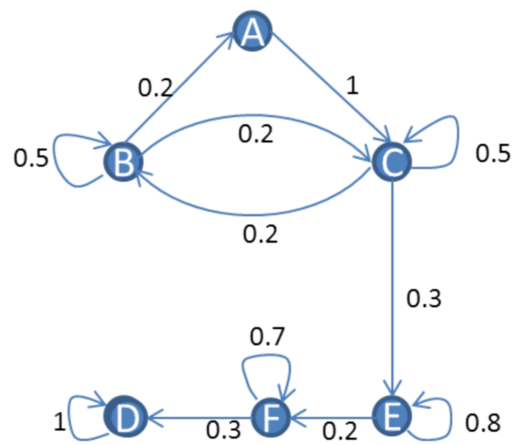
(d) [4] What is the probability that the first three of the four aces are in successive positions in the deck? (You don't have to simplify the expression)

(e) [3] Extra Credit: Suppose now that there are only three aces in the deck of $n \geq 3$ cards. What is the expected number of cards he must turn up to the second ace?



Problem 6 [14] Markov Chains

(a) [4] Identify all recurrent, transient, periodic (with period) and aperiodic states in the Markov Chain in the figure. Identify all the recurrent classes as well.



(b) [5] Suppose $X_0 = E$ in the Markov Chain in this figure. Let T_n be the probability that the first time that the process enters state D is in n time steps. Find the pmf of T_n. HINT: Let e be the amount of time spent in state E and f the amount of time in state F. Write an expression in terms of e and f and try to solve it!

(c) [5] Again using the MC in the figure on this page, suppose that Bob follows the states of this MC by starting in the state E. He collects 2 chocolate chip cookies for every time unit he spends in E and 1 cookie for every time unit he spends in F. He receives no cookies in any other state. What is the expected value of the total number of cookies he collects? What the variance of the total number of cookies he collects?