EE126: Probability and Random Processes

Lecture 9: Independence; Continuous Random Variables

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Logistics

1. HW 4 due date extended to Thursday. HW 5 is out today and due next Tuesday.
2. Calculus Diagnostic Available (see HW folder under Resources in bspace)
3. Midterm covers everything through Thursday. On March 3.

Mean of the Geometric Random Variable

X counts the number of coin tosses to the first head (prob of heads is p). Suppose we have tossed the coin k times with no success. Does that affect the future?
The Geometric RV is **memoryless**:
I.e:

$$p_{X|X\geq k}(k+m) = \frac{p_X(k+m)}{\sum_{i=k}^{\infty} p(1-p)^i} = p(1-p)^{m-1} = p_X(m).$$

Thus $E[X|X>1] = 1 + E[X]$. Let's use this:

$$\begin{aligned} E[X] &= P(X=1)E[X|X=1] + P(X>1)E[X|X>1] \\ &= p + (1-p)(1+E[X]) \\ &= \frac{1}{p} \end{aligned}$$

Variance of the Geometric Random Variable

$$\mathrm{var}(X) = E[X^2] - E[X]^2$$

$$\begin{aligned}
E[X^2] &= P(X=1)E[X^2|X=1] + P(X>1)E[X^2|X>1] \\
&= p + (1-p)E[(1+X)^2] \\
&= p + (1-p)E[X^2] + 1 - p + 2(1-p)\frac{1}{p} \\
&= (1-p)E[X^2] + \frac{2(1-p)}{p} + 1 \\
&= \frac{2-p}{p^2}
\end{aligned}$$

$$var(X) = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$var(X) = \frac{1-p}{p^2}$$

Independence of Random Variables

- Recall that events A and B are independent if $P(A \cap B) = P(A)P(B)$.
- Also, if A, B, C are events: $P(A)P(B)P(C) = P(A \cap B \cap C) \not\Rightarrow A, B, C$ independent.
- X, Y independent if $P_{X,Y}(x, y) = P_X(x)P_Y(y)$, for all x, y.
- In terms of events, $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all x, y means that

$$P(\{X = x\} \cap P(\{Y = y\}), \text{ for all } x, y.$$

Strong statement!

Independence of Random Variables

- If X, Y, Z are rv's: $p_X(x)p_Y(y)p_Z(z) = p_{X,Y,Z}(x, y, z)$ for all $x, y, z \Rightarrow X, Y, Z$ are pair-wise independent.
- Why?
  - For X, Y (or YZ, XZ):

$$\sum_z p_X(x)p_Y(y)p_Z(z) = \sum_z p_{X,Y,Z}(x, y, z)$$

$$p_Y(y)p_Z(z) = p_{X,Y}(x, y).$$

Random variables X, Y, Z are independent if

$$p_{X,Y,Z}(x, y, z) = p_X(x)p_Y(y)p_Z(z)$$

for all x, y, z.

Sums of Random Variables

Suppose X and Y are independent. Let $Z = X + Y$.

$$\begin{aligned} p_Z(z) &= \sum_x \sum_{y:x+y=z} p_{X,Y}(x,y) = \sum_x p_{X,Y}(x, z-x) \\ &= \sum_x p_X(x) p_Y(z-x) = \sum_y p_x(z-y) p_Y(y) \end{aligned}$$

Thus Z=X*Y (X convolved with Y)

Example: $p_X(x) = \frac{1}{4}$ $x = 0, 1, 2, 3$. Y is independent from X and $p_Y(y) = \frac{1}{2}$ $y = 0, 1$ If $Z = X + Y$, What is $p_Z(z)$?

Expected Value and Variance

If X, Y are independent:

$$E[XY] = \sum_{x,y} xy p_{X,Y}(x,y) = \sum_{x,y} x p_X(x) y p_Y(y)$$

$$E[XY] = E[X]E[Y]$$

Also,

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) \Rightarrow \sigma_{X+Y} = \sqrt{\mathrm{var}(X) + \mathrm{var}(Y)}.$$

$$\mathrm{var}(X+Y) = E[(X+Y)^2] - E[X+Y]^2 \qquad (E(X)+E(Y))^2$$

$$= E[X^2 + 2XY + Y^2] - E(X)^2 - E(Y)^2 - 2E[X]E[Y]$$

$$= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - E[Y]^2 - 2E[X]E[Y]$$

$$= E[X^2] - (E[X])^2 + E[Y^2] - E(Y)^2 + 2(E[XY] - E[X]E[Y])$$

$$\mathrm{var}(X) + \mathrm{var}(Y) + 2\,\text{covariance}(X,Y)$$

Example: Variance of a Binomial Distribution

For N trials and probability p of success we know that

$$X = \sum_{i=1}^{n} B_i$$

where the Bernoulli are independently and identically distributed (iid) rvs with probability of success p.

$$\mathrm{var}(X) = \sum_{i=1}^{n} \mathrm{var}(B_i) = n\mathrm{var}(B_1)$$

$$\mathrm{var}(B_1) = (1-p)^2 p + p^2(1-p) = (1-p)((1-p)p + p^2) = (1-p)p.$$

$$\mathrm{var}(X) = np(1-p)$$

For $p << 1$, $var(X) \approx E[X] = np$.

Example: Fireman Hiring Problem

Equal number of houses in two areas in a city:
Area A: Prob of a house catching fire on any given day = p;
Area B: Half the houses have prob of catching fire $\frac{p}{2}$ and the other half $\frac{3}{2}p$ ($p < \frac{2}{3}$).
$X_A(X_B)$: Number of houses that catch fire on a given night on a given day in area A (B)
Need to hire firemen in proportion to $E[X_A] + \sigma_{X_A}$ in area A and in proportion to $E[X_B] + \sigma_{X_B}$ in area B. Which area will need more firemen?

$E[X_A] = E[X_B] = np$.

$$var(X_A) = np(1 - p).$$

$$\mathrm{var}(X_B) = \frac{n}{2}\frac{p}{2}(1 - \frac{p}{2}) + \frac{n}{2}\frac{3p}{2}(1 - \frac{3p}{2}) = np(1 - 1.25p)$$

So the homogeneous area needs more firemen!

Continuous Random Variables

In many examples, a continuous sample space is more natural than a discrete one:

1. Measuring time: Jack and Jill are supposed to meet at 4pm, but each will arrive with a delay between 0 and 0.5 hours, with all pairs of delays equally likely. The first to arrive will wait for 10 mins and then leave. What is the probability that they meet?
2. Measuring distance: Two spots are randomly chosen along the length of a stick of length 1 foot. The stick is cut each of these points. What is the probability the three pieces can form a triangle?

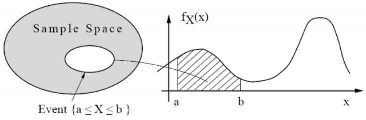
How do we define probabilities?

Issues with Continuous Sample Spaces

- In the discrete case any subset of the basic of the basic outcomes is an event.
- In the continuous case, not all subsets of the sample space are allowable as events. One can come up with subsets that are so strange that there is no way to assign probabilities to them!

How can this be? The set has to be quite strange. Example in HW...

Continuous Random Variables



X is continuous if

1. there is a non-negative function f_X such that

$$P(X \in B) = \int_B f_X(x)dx$$

is well defined for every subset B of the real line;

2. $$\int_{-\infty}^{\infty} f_X(x) = 1.$$

- $P(a \leq X \leq b) = \int_a^b f_X(x)dx$
- $P(X = a) = 0$
- $P(X < a) = P(X \leq a)$

Thus we are limiting events in some way...

The Probability Density Function

$$P([x, x+\delta]) = \int_x^{x+\delta} f_X(t)dt \approx f_X(x)\delta$$

$$f_X(x) \approx \frac{P([x, x+\delta])}{\delta}$$

Thus $f_X(x)$ is **not the probability** of any event!
Example:

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 < x \leq 1; \\ 0, & \text{o.w.} \end{cases}$$

$f_X(x)$ can be arbitrarily large but $\int_0^1 \frac{1}{2\sqrt{x}} dx = 1$.

Example: Uniform Distribution

Alice is equally likely to show up for a meeting at any time in the interval between 4 and 4:30pm.
Measure time in minutes from 4PM: $t = 0$ is 4PM and $t = 30$ is 4:30PM.
Then $f_T(t) = a$ for $t \in [0, 30]$ and since $\int_{t=0}^{30} a \, dt = 1$, we have $a = \frac{1}{30}$.

Uniform Distribution

Uniform distributions make calculations easier. Find areas:
Alice/Bob Problem (they only wait 10 mins)
Total area = 900. Event area = 500
Prob(Alice and Bob meet)= $\frac{5}{9}$.
In general, if their window of arrival is W mins and they are willing to wait only T minutes:

$$\frac{W^2 - (W - T)^2}{W^2}$$

$$Prob(theymeet) = \frac{2WT - T^2}{W^2}.$$

PDFs

Alice has two ways to get to a concert. With probability $1/3$ her friend, Bob will drive her there in time uniformly distributed over $[30, 45]$ minutes. With probability $2/3$ she will have to take the bus which takes time distributed with density $\alpha t - \beta$ for $t \in [50, 70]$. What are α, β and the density function for the time it will take Alice to get to the concert?

$$f_T(t) = \begin{cases} \frac{1}{45}, & t \in [30, 45]; \\ \frac{t}{300} - \frac{1}{6}, & t \in [50, 70]; \\ 0, & \text{o.w.} \end{cases}$$

Expected Value and Variance

Straightforward extension from discrete case:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

(as long as the integral converges absolutely)

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$var(X) = \int_{-\infty}^{\infty} (x - E[x])^2 f_X(x) dx$$

$$var(X) = E[X^2] - E[X]^2$$

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

$$var(aX) = a^2 var(X)$$

etc.

Mean and Variance of Uniform RV

X ~Uniform in $[a, b]$:

$$E[X] = \int_a^b x \, \frac{1}{b-a} dx = \frac{1}{b-a} \frac{1}{2} x^2 \Big|_a^b$$

$$E[X] = \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \, \frac{1}{b-a} dx = \frac{1}{b-a} \frac{1}{3} x^3 \Big|_a^b$$

$$(b^3 - a^3) = (b-a)(b^2 + ab + a^2)$$

$$E[X^2] = \frac{a^2 + ab + b^2}{3}$$

$$\mathrm{var}(X) = E[x^2] - E[X]^2 = \frac{(b-a)^2}{12}$$

Example: Laplace Distribution

X: $f_x(x) = ke^{-\lambda|x|}$ for $\lambda > 0$ and $x \in \Re$.
For what value of k is this a valid density function?

1. For $x \geq 0$:

$$k \int_0^\infty e^{-\lambda x} dx = -\frac{k}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{k}{\lambda}$$

2. For $x < 0$:

$$k \int_{-\infty}^0 e^{\lambda x} dx = \frac{k}{\lambda} e^{\lambda x}\Big|_{-\infty}^0 = \frac{k}{\lambda}$$

$$\int_{-\infty}^\infty f_x(x) dx = \frac{2k}{\lambda}$$

So

$$k = \frac{\lambda}{2}.$$

What are $E[X]$ and $var(X)$?

Mean and Variance of Laplace Distribution

$E[X]$ is likely to be 0 since the distribution is symmetric around zero. Still need to check!
$E[X] = 0$ only if $\frac{\lambda}{2}\int_0^\infty xe^{-\lambda x}dx$ is finite.

$$(uv)' = u'v + v'u \Rightarrow u'v = (uv)' - uv' \Rightarrow \int vdu = uv - \int udv$$

Let $du = e^{-\lambda x}$, $v = x$:

$$\begin{aligned}\int_0^\infty x\lambda e^{-\lambda x}dx &= |_0^\infty -xe^{-\lambda x} + \int e^{-\lambda x}dx \\ &= 0 - \frac{e^{-\lambda x}}{\lambda}|_0^\infty \\ &= \frac{1}{\lambda}\end{aligned}$$

Therefore, $E[X] = 0$.

Mean and Variance of Laplace Distribution

$$E[X^2] = \frac{\lambda}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|\lambda} dx = \lambda \int_0^{\infty} x^2 e^{-x\lambda} dx$$

$$(uv)' = u'v + v'u \Rightarrow u'v = (uv)' - uv' \Rightarrow \int vdu = uv - \int udv$$

Let $du = e^{-\lambda x}$, $v = x^2$:

$$\lambda \int_0^{\infty} x^2 e^{-x\lambda} dx = \lambda\Big|_0^{\infty} -\frac{1}{\lambda}x^2 e^{-x\lambda} + \frac{2}{\lambda}\int xe^{-\lambda x} dx$$

So

$$E[X^2] = 2\int_0^{\infty} xe^{-\lambda x} dx = \frac{2}{\lambda}\int_0^{\infty} x\lambda e^{-\lambda x} dx$$

We just evaluated the integral...

$$E[X^2] = \frac{2}{\lambda}\frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$var(X) = E[X^2] = \frac{2}{\lambda^2}$$

Example: Share price

The price of XYZ share changes daily according to a Laplace RV with $\lambda = 10$. What is the probability that it will go up by no more than greater than .50 and go down by no more than .25

Let X be the amount the price changes in one day.

We want

$$\begin{aligned}
Pr(.5 \leq X \leq 1) &= 5\int_{-.25}^{.5} e^{-10|x|}dx \\
&= 5(\int_{-.25}^{0} e^{10x}dx + \int_{0}^{0.5} e^{-10x}dx) \\
&= 5(\frac{1}{10}\Big|_{-.25}^{0} e^{10x} - \frac{1}{10}\Big|_{0}^{0.5} e^{-10x}) \\
&= 5(\frac{1}{10}(1 - e^{-2.5}) - \frac{1}{10}(e^{-5} - 1)) \\
&= \frac{1}{2}(2 - e^{-2.5} - e^{-5}) = .955
\end{aligned}$$