EE126: Probability and Random Processes

Lecture 5: Discrete Random Variables

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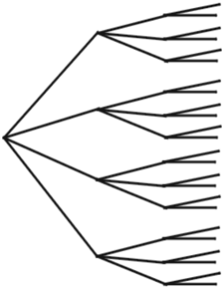
Logistics

1. Pre-Reading
2. Asking Questions
3. Piazzza
4. ?

Counting Principle: Sequential Interpretation

Counting Principle

There are r experiments; experiment i has n_i outcomes. The total number of outcomes is $n_1 n_2 \dots n_r$.



How many subsets of $\{1, 2, \dots, n\}$? 2^n.

k-Permutations

How many different ways are there to list any k of n things?

k-Permutations

The number of k-Permutations $(k = 0, 1, 2, ..., n)$ of n objects is

$$n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$$

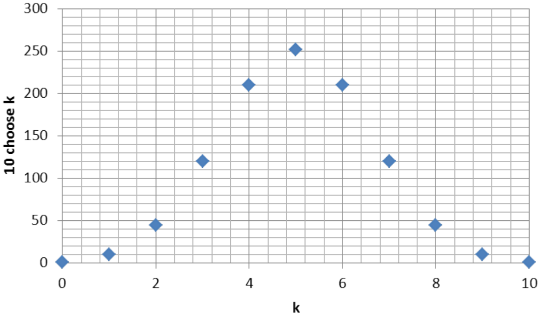
If $k = n$ then this is called a permutation and there are $n!$ ways.
By convention, $0! = 1$.

Definition

Combinations

Let $\binom{n}{k}$ be the number of ways to choose $k \leq n$ objects from n. Then

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$



$$\binom{n}{k} = \binom{n}{n-k}$$

Generalizing Combinations to k-partitions

Partitioning n distinct objects into groups of $n_1, \ldots, n_r$

In general if there are n objects, n_1 of one kind are alike, n_2 of another kind are alike...,n_r are alike, then the number of permutations is

$$\frac{n!}{n_1!n_2!\ldots n_r!}.$$

Partitions

How many ways are there to divide n identical objects into k groups?

Partitioning n identical objects

There are $\binom{n+k-1}{n}$ distinct non negative integer-valued vectors $(x_1, ..., x_k)$ satisfying

$$x_1 + x_2 + \ldots + x_n = n.$$

Back to original question...

13 people are randomly divided into 3 groups.
Prob(two of the groups have the same number of people)=

$$\frac{\text{number of ways to get groups with two same size}}{\text{number of ways to divide into 3 groups}}$$

$$\frac{3\sum_{k=0}^{6}\frac{13!}{k!k!(13-2k)!}}{3^{13}}$$

A	B	C	#groups	X 3
0	0	13	1	3
1	1	11	156	468
2	2	9	4290	12870
3	3	7	34320	102960
4	4	5	90090	270270
5	5	3	72072	216216
6	6	1	12012	36036
	Total # ways with 2 same size =			638823
	Total # ways to group into 3 =			1594323
				0.400686

From Events to Random Variables

- So far we have been dealing with Events which are subsets of the Sample Space.
- Random variables associate a real number with each event
- Examples:
  - the rv $X = i$ if the throw of a die is $\{i\}$
  - X^2: the square of the value of a die throw.
  - the rv $X = 1$ if a particular event occurs and $X = 0$ otherwise

$$X = \begin{cases} 1, & \text{if two throws of a die includes a } 6; \\ 0, & \text{otherwise.} \end{cases}$$

Random variables allow us to calculate averages, etc

Example: Two 4-sided Dice

M_k: the event that the minimum is k.
The random variable M is **equal** to the value of the minimum.
Given a basic outcome (e.g. (2,3)), M assumes a specific value (e.g. 2).
Let M : Minimum of the two die

	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

Each basic outcome of the experiment maps to **exactly one value** of the random variable.
A random variable is a function from the outcomes of an experiment to the set of real numbers.

2 Dice Continue

Define $E = 1$ if one of the rolls is even and $E = 0$ if one of the rolls is odd.
This is not a random variable. Why?
The basic outcome $(0, 1)$ maps to more than value of E.
The domain of a random variable **must be a partition** of the sample space.
Example: Define $E = 1$ if one of the rolls is even and $E = 0$ otherwise.

Discrete and Continuous RVs

Discrete: The rv can take on a finite or countably infinite number of values.

Continuous: The rv takes on an uncountably infinite number of values.

Example: A point is chosen at random on the line $[0, 1]$.

- Continuous r.v. a: the value of the point chosen
- Discrete r.v. b: $b = 1$ if $a \geq 0.5$ and $b = 0$ otherwise.

For now, we will only focus on DISCRETE random variables.

Probability Mass Functions

Defines the probability for each possible value of a discrete random variable.

Example: X: the number of heads when a fair coin is tossed three times.

$$p_X(x) = \begin{cases} 1/8, & \text{x=0;} \\ 3/8, & \text{x=1;} \\ 3/8, & \text{x=2;} \\ 1/8, & \text{x=3.} \end{cases}$$

Generally, for each possible value x of X:

1. Collect all the possible outcomes that give rise to $\{X = x\}$.
2. Add their probabilities to obtain $p_X(x)$.

Example: Two 4-sided Dice

M = the minimum of the two throws.

	1	2	3	4
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

$\Rightarrow$

	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

The probability mass function is:

$$p_M(m) = \begin{cases} \quad , & \text{m=1;} \\ \quad , & \text{m=2;} \\ \quad , & \text{m=3;} \\ \quad , & \text{m=4.} \end{cases}$$

Probability Mass Function

For any random variable, X,

$$\sum_x p_X(x) = 1.$$

Why?

- Since X is a function from Ω each basic outcome corresponds to **exactly one** value of X.
- Summing over all values of X, is like summing over all the basic outcomes.

Example: Chess Match

Anand plays Kasparov: the first player to win a match wins the match. The match is drawn if there are 10 successive draws. Prob(Anand wins a game)=.3, P(Kasparov wins a game)=.4, P(draw)=.3.
What is the pmf of the duration of the match L?

- $L = 10$ if and only if the first 9 matches are drawn. $P(L = 10) = .3^9$.
- $L = k < 10$ if $k - 1$ matches are drawn and the k^{th} match is not drawn. $P(L = k) = 0.3^{k-1}0.7$, $k = 1, 2, ..., 9$.

What is the probability that Anand wins the match?

- There can $k = 0, 1, .., 9$ draws but then Anand must win a game. Prob(Anand wins) $= \sum_{k=0}^{9} 0.3^k 0.3 = 0.428568898$.

Geometric Random Variable

The Geometric random variable **counts the time to first success**. The probability of success is p and we keep performing independent trials of an experiment until the first success occurs.

$$p_X(k) = (1-p)^{k-1}p$$

- X: number of independent coin tosses until the first head.
- X: number of rolls of a die until "5"
- X: number of lottery tickets you have to buy until you win.

Binomial Random Variable

$X =$ number of heads in N coin tosses.

1. There are $\binom{N}{k}$ ways to get k Heads.
2. Any given N-toss sequence with k heads has probability $p^k(1-p)^{N-k}$ where p is P(Heads).

$$p_X(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

Bernoulli RV: $B = 1$ if a coin flips head and $B = 0$ otherwise
Flip a coin N times, then there are N values: $B_1, B_2, ..., B_N$. What is $\sum_i B_i$? $X = \sum_i B_i$.
A Binomial Random Variable counts successes.

Poisson Random Variable

Counting Successes with a Binomial rv can be computationally difficult:

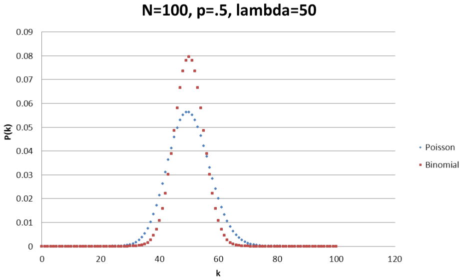
1. Count misprints in a book. N is huge and p is very small.
2. What is the probability that exactly k out 500 randomly chosen people will share a birthday on New Year's Day?
3. A factory produces defective screws with prob 0.015. What is the prob that a box of 100 screws has k defective items?

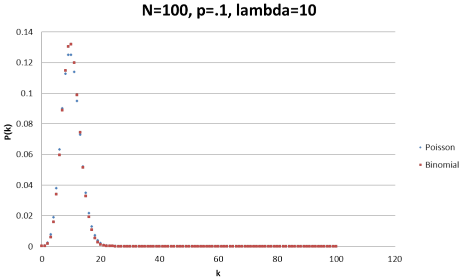
When N is large and p is small an excellent approximation is:

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

where $\lambda = Np > 0$.

Approximation of Binomial by Poisson





N=100,p=.01,lambda=1

	Poisson	Binomial
0	0.3678794	0.366032
1	0.3678794	0.36973
2	0.1839397	0.184865
3	0.0613132	0.060999
4	0.0153283	0.014942
5	0.0030657	0.002898
6	0.0005109	0.000463
7	7.299E-05	6.29E-05
8	9.124E-06	7.38E-06
9	1.014E-06	7.62E-07
10	1.014E-07	7.01E-08

Why?

Fact: $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$.

Start with Binomial...set $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$
\begin{aligned}
p_X(k) &= \frac{n!}{(n-k)!k!}p^k(1-p)^{n-k} \\
&= \frac{n(n-1)\ldots(n-k+1)}{n^k}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^{n-k} \\
&= \frac{n}{n}\frac{n-1}{n}\ldots\frac{n-k+1}{n}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^{-k}(1-\frac{\lambda}{n})^n
\end{aligned}
$$

As $n \to \infty$, $p_X(k) \to e^{-\lambda}\frac{\lambda}{k!}$ when $k << n$.

Functions of a Random Variable

Suppose $p_X(x) = \frac{1}{5}$, $x = -2, -1, 0, 1, 2$.

Define $Y = |X| - X$. What is $p_Y(y)$?

x	y
-2	4
-1	2
0	0
1	0
2	0

$$
\begin{aligned}
p_Y(4) &= p_X(-2) \\
p_Y(2) &= p_X(-1) \\
p_Y(0) &= p_X(0) + p_X(1) + p_X(2)
\end{aligned}
$$

So

$$
p_Y(y) = \begin{cases} \frac{1}{5}, & \text{y=2,4;} \\ \frac{3}{5}, & \text{y=0;} \\ 0, & \text{o.w.} \end{cases}
$$

Functions of Random Variables

Given a random variable X and a function $Y = g(X)$:

$$p_Y(y) = \sum_{\{x : g(x) = y\}} p_X(x).$$