EE126: Probability and Random Processes

Lecture 4: Counting

Abhay Parekh

UC Berkeley

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Logistics

1. Recitations: Arash on deck
2. Annotated Versions of LNs available

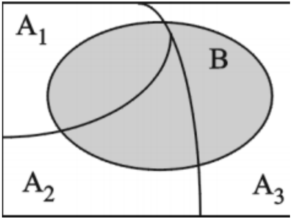
Review

Conditional Probability

If $P(B) \neq 0$ then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for any event A.



Multiplication Rule

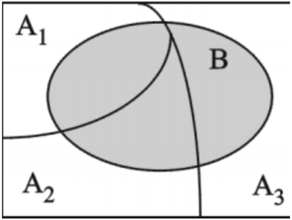
Assuming $A_1, ..., A_n$ have positive probability:

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)...P(A_n|\bigcap_{i=1}^{n-1} A_i)$$

Total Probability

$A_1, ..., A_N$ are mutually exclusive, non-empty events and that $\sum_{i=1}^{N} P(A_i) = 1$.

For any event B: Each of the base outcomes that comprise B must be in exactly one of $A_1, A_2, ..., A_N$. Suppose we can calculate $P(B|A_i)$ for all i. Then we can find $P(B)$.



Total Probability Theorem

$$P(B) = P(A_1 \cap B) + \ldots + P(A_N \cap B)$$
$$= P(A_1)P(B|A_1) + \ldots + P(A_N)P(B|A_N)$$

Bayes Rule

$A_1, ..., A_N$ partition Ω. We know the $P(A_i)$. Also, we know $P(B|A_i)$ for all i. Want $P(A_i|B)$

Example: An Xray has a shade (event B) which might be due to a malignant tumor (A_1), benign tumor (A_2), or other (A_3). We know $P(B|A_i)$, but what we care about is $P(A_i|B)$.

Bayes Rule

$$
\begin{aligned}
P(A_i|B) &= \frac{P(A_i)P(B|A_i)}{P(B)} \\
&= \frac{P(A_i)P(B|A_i)}{\sum_j^N P(A_j)P(B|A_j)}
\end{aligned}
$$

Independence

- Two events: $P(A \cap B) = P(A)P(B)$, $P(A|B) = P(A)$.
  - A, B independent $\Rightarrow B, A$, A^c, B , A^c, B^c independent.
- Conditional independence: $A|C$ and $B|C$ independent, i.e. $P(A \cap B|C) = P(A|C)P(B|C)$.
  - Two conditionally independent events may be (unconditionally) dependent
  - Two conditionally dependent events may be (unconditionally) independent
- General Definition:
  - A collection of events $A_1, ..., A_n$ are independent if

$$P(\cap_{i \in S} A_i) = \Pi_{i \in S} P(A_i)$$

for every subset S of $\{1, 2, \ldots, n\}$.

Discrete Uniform Law

Suppose all the base outcomes of an experiment are equally likely

Given any event A:

$$P(A) = \frac{\text{Number of base outcomes in } A}{\text{Total number of base outcomes}} = \frac{|A|}{|\Omega|}.$$

So just count.
Sometimes easy: Example: Toss two dice; Prob(both roll to an identical number)
But sometimes not so easy: Example: 13 people are randomly divided into 3 groups. Prob(two of the groups have the same number of people)= $\ddot\smile$
Need some counting machinery!

The Counting Principle

Suppose there are 2 experiments to be performed. Experiment 1 can result in m different outcomes, and experiment 2 can result in n different outcomes, then together there are mn outcomes.

(1,1)	(1,2)	...	(1,n)
(2,1)	(2,2)	...	(2,n)
...			
(m,1)	(m,2)	...	(m,n)

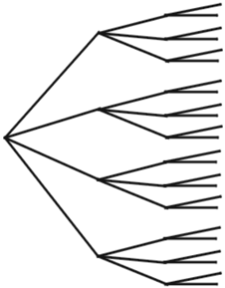
Counting Principle

There are r experiments; experiment i has n_i outcomes. The total number of outcomes is $n_1 n_2 \dots n_r$.

Counting Principle: Sequential Interpretation

Counting Principle

There are r experiments; experiment i has n_i outcomes. The total number of outcomes is $n_1 n_2 \ldots n_r$.



How many subsets of $\{1, 2, \ldots, n\}$? 2^n.

k-Permutations

How many different ways are there to list any k of n things?

- n ways to pick position 1 of the list
- $n-1$ ways to pick position 2
- $n-k+1$ to pick position k

From the counting principle: $n(n-1)...(n-k+1)$ ways.

k-Permutations

The number of k-Permutations $(k=0,1,2,...,n)$ of n objects is

$$n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$$

If $k=n$ then this is called a permutation and there are $n!$ ways.
By convention, $0! = 1$.

Examples

42 students in a probability class: 32 men and 10 women.

1. How many ways to rank the first five? $\frac{42!}{37!}$

2. Prob(they are ranked in increasing alphabetical order) $= \frac{1}{42!}$.
3. Prob(men are ranked in increasing alphabetical order)? $\frac{1}{32!}$

4. Men and women are ranked separately: Prob(men and women are each ranked in increasing alphabetical order)? $\frac{1}{10!}\frac{1}{32!}$

Examples

1. Prob(six rolls of a die result in 6 different numbers)$=\frac{6!}{6^6}$

2. How many ways are there to seat n people around a round table?$(n-1)!$.

Permutations with alike objects

Permutations of the word "SEEKS"

1. List the $5! = 120$ ways assuming all the letters are distinguishable, i.e. $S_1 E_1 E_2 K_1 S_2$.
2. Consider any instance, e.g. $S_2 E_2 E_1 S_1 K_1$:
   - Since the S's are actually indistinguishable, 2! times the actual number. Same for the E's.
   - Thus, there are $2!2! = 4$ times the correct number of instances of $S_2 E_2 E_1 S_2 K_1$.
3. Thus there $\frac{120}{4} = 30$ ways to list "SEEKS".

Permutations with Alike Objects

In general if there are n objects, n_1 of one kind are alike, n_2 of another kind are alike..., n_r are alike, then the number of permutations is

$$\frac{n!}{n_1! n_2! \ldots n_r!}.$$

Combinations

We have n objects: 1,2,...,n.
Want to select $k \leq n$ of them.
We don't want to list them, just select k. How many different ways?

1. Assume that the order of the k selected matters:

$$(n)(n-1)(n-2)...(n-k+1)$$

2. There are $k!$ ways to list k objects, so each each listing generated above is repeated $k!$ times. Thus, there are

$$\frac{n(n-1)...(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

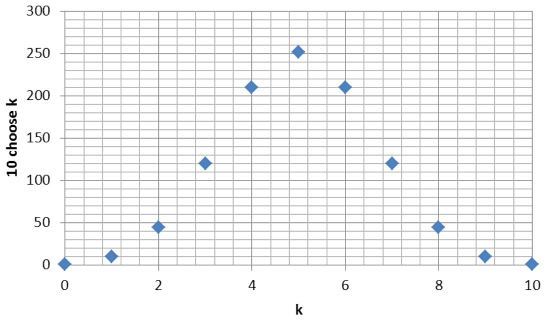
ways to select k objects from n objects.

Definition

Combinations

Let $\binom{n}{k}$ be the number of ways to choose $k \leq n$ objects from n. Then

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$



$$\binom{n}{k} = \binom{n}{n-k}$$

Examples

1. Three students out of 10 are picked for a committee. How many ways? $\binom{10}{3}$
2. Ten students at the rec center divide themselves into 2 teams of 5 each. How many ways? $\binom{10}{5}$
3. Ten students need to divided into an A-Team and a B-Team of 5 each. How many ways? $2 * \binom{10}{5}$

Binomial Distribution

In N coin tosses how many ways to get k Heads? Any k of the tosses could be heads so $\binom{N}{k}$.
Any given N-toss sequence with k heads has probability $p^k(1-p)^{N-k}$ where p is P(Heads).

Binomial Distribution

The probability of getting k heads in N tosses when P(H)=p is

$$\binom{N}{k} p^k (1-p)^{N-k}.$$

Do the probabilities sum up to 1?

Binomial Theorem

- $(a+b)^2 = a^2 + b^2 + 2ab$
- $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$
- $(a+b)^4 = a^4 + b^4 + 4a^3b + 4ab^3 + 6a^2b^2$
- $\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$

  Back to the binomial distribution:

$$\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = (p + (1-p))^N = 1$$

Generalizing Combinations to k-partitions

How many ways can divide n objects into r groups with n_i objects in the i^{th} group?
Example: Ten students need to be divided into three teams of sizes 3,3,4. How many ways?

$$\begin{aligned}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_r}{n_r} &= \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{n_k!}{n_r!1} \\ &= \frac{n!}{n_1!n_2!\dots n_r!}\end{aligned}$$

Partitioning n distinct objects into groups of $n_1, \ldots, n_r$

In general if there are n objects, n_1 of one kind are alike, n_2 of another kind are alike...,n_r are alike, then the number of permutations is

$$\frac{n!}{n_1!n_2!...n_r!}.$$

Generalizing Combinations to k-partitions

The formula is identical to the one for permutations with like objects!
Recall Permutations of the word "SEEKS"

- The distinct letters tag the positions $\{1, 2, 3, 4, 5\}$ with a group "name".
- Example: $EESSK \Rightarrow 1, 2$ in group E, $3, 4$ in group S and 5 in group K.
- Number of distinct ways of writing $SEEKS =$ Number of ways of dividing 5 things into three groups of sizes 2,2 and 1.
- Combinations are a special case: $\binom{n}{k}$ is the number of permutations of a n-word with k A's and $n - k$ B's.

Partitions

How many ways are there to divide n identical objects into k groups?

Example: $n = 4$ $k = 2$: $\{(4,0),(3,1),(2,2),(1,3),(0,4)\} \Rightarrow 5$.

Code each way with n 1's and $k-1$ 0's. For example, if $n = 5, k = 3$, then

$$(1,2,2) \Leftrightarrow (1,0,1,1,0,1,1)$$

There are $\binom{n+k-1}{n}$ ways of picking the boolean vector.

Partitioning n identical objects

There are $\binom{n+k-1}{n}$ distinct non negative integer-valued vectors $(x_1, ..., x_k)$ satisfying

$$x_1 + x_2 + \ldots + x_n = n.$$

Back to original question...

13 people are randomly divided into 3 groups.
Prob(two of the groups have the same number of people)=