EE126: Probability and Random Processes

Lecture 3: Independence

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HW Due

Please make sure you hand it end by the end of class. No late HWs accepted.
(HW #2 is already posted)
We have started using Piazzza and questions are being posted there. Please register.
Syllabus updated on bspace. First midterm 3/3.

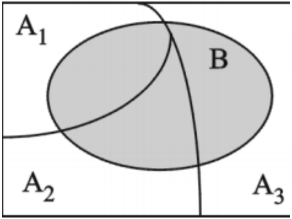
Review

Conditional Probability

If $P(B) \neq 0$ then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for any event A.



Multiplication Rule

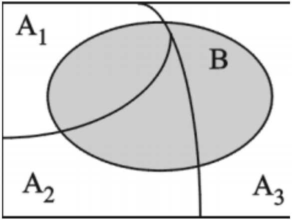
Assuming $A_1, ..., A_n$ have positive probability:

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)...P(A_n|\bigcap_{i=1}^{n-1} A_i)$$

Total Probability

$A_1, ..., A_N$ are mutually exclusive, non-empty events and that $\sum_{i=1}^{N} P(A_i) = 1$.

For any event B: Each of the base outcomes that comprise B must be in exactly one of $A_1, A_2, ..., A_N$. Suppose we can calculate $P(B|A_i)$ for all i. Then we can find $P(B)$.



Total Probability Theorem

$$\begin{aligned} P(B) &= P(A_1 \cap B) + \ldots + P(A_N \cap B) \\ &= P(A_1)P(B|A_1) + \ldots + P(A_N)P(B|A_N) \end{aligned}$$

Bayes Rule

$A_1, ..., A_N$ partition Ω. We know the $P(A_i)$. Also, we know $P(B|A_i)$ for all i. Want $P(A_i|B)$

Example: An Xray has a shade (event B) which might be due to a malignant tumor (A_1), benign tumor (A_2), or other (A_3). We know $P(B|A_i)$, but what we care about is $P(A_i|B)$.

Bayes Rule

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j^N P(A_j)P(B|A_j)}$$

Keeping Up

Alice is taking EE126 and at the end of each week she can be "up-to-date" or behind.

- If she is behind in a given week, she will behind in the next week with prob 0.6.
- If she is up-to-date in a given week she will be up-to-date in the next week with prob 0.8.
- What is the probability that she is up-to-date after three weeks?

U_i: she is up-to-date after week i weeks. B_i: she is behind after i weeks.

$$
\begin{aligned}
P(U_3) &= P(U_3, U_2) + P(U_3, B_2) = \\
&=
\end{aligned}
$$

$P(U_2) = P(U_1)P(U_2|U_1) + P(B_1)P(U_2|B_1) =$
$P(B_2) = P(U_1)P(B_2|U_1) + P(B_1)P(B_2|B_1) = P(U_1)0.2 + P(B_1)0.6$
Now $P(B_1) =$ and $P(U_1) =$.

Keeping Up

Alice is taking EE126 and at the end of each week she can be "up-to-date" or behind.

- If she is behind in a given week, she will behind in the next week with prob 0.6.
- If she is up-to-date in a given week she will be up-to-date in the next week with prob 0.8.
- What is the probability that she is up-to-date after three weeks?

U_i: she is up-to-date after week i weeks. B_i: she is behind after i weeks.

$P(U_3) = P(U_2)0.8 + P(B_2)0.4$
$P(U_2) = P(U_1)0.8 + P(B_1)0.4$
$P(B_2) = P(U_1)0.2 + P(B_1)0.6$
Now $P(B_1) = 0.2$ and $P(U_1) = 0.8$.
$P(U_2) =$
$P(B_2) =$
$P(U_3) =$

Independence

Two events are independent if the occurrence of one provides no information about the occurrence of the other. I.e.,

$$P(A|B) = P(A)$$

$\Rightarrow$

Independence of Events

Events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

If A is independent of B then B is independent of A

Examples: Tosses of a coin; rolls of a die.

Check...

Toss a coin three times.

1. A: H on the first toss, B: tails on the third toss.
   $P(A) = P(B) = \frac{1}{2} \Rightarrow P(A)P(B) = \frac{1}{4}$
   $P(A \cap B) = P(\{HTT, HHT\}) = \frac{2}{8}$.
   A and B are independent.
2. C: First two tosses H.
   $P(C) = \frac{2}{8}$. $P(A \cap C) = \frac{2}{8}$.
   A and C are not independent.

Independent Events

- When the event A is conditioned on B, none of the basic outcomes in $A \cap B^c$ can occur. $\Rightarrow$ its probability "shrinks".
- Because the new sample space is B, we normalize by $P(B) \Rightarrow$ probability "expands"
- Since $P(A|B) = P(A)$ when A and B are independent:

$$\text{(amount of shrinkage)} \times \text{(amount of expansion)} = 1$$

$$\frac{P(A \cap B)}{P(A)} \frac{1}{P(B)} = 1.$$

Since $P(A \cap B) = P(A) - P(A \cap B^c)$:

$$P(A)(1 - P(B)) = P(A \cap B^c)$$

$$P(A)P(B^c) = P(A \cap B^c)$$

A and B^c are independent as well.

Disjoint Events

Disjoint Events, A, B: $P(A \cap B) = 0$
Can only be independent if either $P(A) = 0$ or $P(B) = 0$.
The base outcomes of any experiment are disjoint $\Rightarrow$ they are always dependent.

Example: 2 Dice

- A: Sum of the two dice is 6
- B: The first die is a 4
- $P(A \cap B) =$
- $P(A) = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} = \frac{5}{36}$
- $P(B) = \frac{1}{6}$
- A and B are not independent.

- A^*: Sum of the two dice is 7
- B: The first die is a 4
- $P(A^* \cap B) = P(\{4,3\}) = \frac{1}{36}$.
- $P(A^*) =$
- $P(B) = \frac{1}{6}$
- A^* and B are independent.

Why the difference??

Conditional Independence

Since $A|C$ and $B|C$ are events it is possible for them to independent:

$$P(A \cap B|C) = P(A|C)P(B|C).$$

When C has known to have occurred, the occurrence of A does not give any information about the occurrence of B.

1. Two events that are unconditionally dependent may be conditionally independent.
2. Two events that are independent be conditionally dependent.

Conditionally independent events may be dependent

Example: Two rolls of a 4-sided die. A: First roll is odd, or the product of the two rolls is 2; B: Second roll is odd; C: product of the rolls is odd.

- If C occurs then, both A and B must occur.
- $P(A|C) = P(B|C) = P(A \cap B|C) = 1 \Rightarrow A|C$ and $B|C$ are independent.
- $P(A) =$
- $P(B) =$, $P(C) =$
- $A \cap B = \{(1,1),(1,3),(3,1),(3,3),(2,1)\} \Rightarrow P(A \cap B) = \frac{5}{16}$
- $$P(A)P(B) = \frac{9}{32} \neq P(A \cap B).$$
- A and B are dependent.

Meta-Coin

Two indistinguishable coins: one lands heads with prob. 0.9, and the other lands on heads with prob .1.
Experiment: Pick one coin at random and flip it two times. H_i: ith flip is H. Are H_1 and H_2 independent? What is $P(H_2|H_1)$?
$+$: the coin that lands H with prob 0.9 is picked.
$P(H_2|H_1 \cap +) = 0.9, \quad P(H_2|H_1 \cap +^c) = 0.1 \Rightarrow H_1, H_2$ conditionally independent given F.
$P(H_1) = P(+)P(H_1|+) + P(+^c)P(H_1|+^c) =$
$P(H_2) =$
$P(H_1 \cap H_2|+) = 0.81, \quad P(H_1 \cap H_2|+^c) = 0.01$
$P(H_1 \cap H_2) = P(+)P(H_1 \cap H_2|+) + P(+^c)P(H_1 \cap H_2|+^c).$

H_1 and H_2 are not independent.

$$P(H_2|H_1) =$$

Independent events can be conditionally dependent

Example: Two tosses of a fair coin. Three events: H_1: first toss is a head; H_2: second toss is a head; D: both tosses are different
H_1 and H_2 are independent. What about $H_1|D$ and $H_2|D$?
$P(H_1 \cap H_2|D) = 0$.
$P(D) = P(\{HT, TH\}) = 0.5$
$P(H_1|D) = P(\{HT\})/P(D) = 0.5$ and $P(H_2|D) = 0.5$ as well. So

$$P(H_1 \cap H_2|D) \neq P(H_1|D)P(H_2|D).$$

Pairwise independence $\not\Rightarrow$ Joint independence

Example: Two tosses of a fair coin. Three events: H_1: first toss is a head; H_2: second toss is a head; D: both tosses are different

- H_1 and H_2 are independent.
- H_1 and D are independent
- H_2 and D are independent.
- $P(H_1)P(H_2)P(D) = \frac{1}{8} \neq P(H_1 \cap H_2 \cap D) = 0$.

H_1, H_2, D are not independent.

Joint Independence of more than 2 events

Example: Two rolls of a fair die. A: $\{\text{1st roll is } 1,2,3\}$; $B:$ $\{\text{1st roll is } 3,4,5\}$ $C:$ sum of both rolls is 9

$P(A) = P(B) = 0.25$;

$$P(C) = \frac{|\{(3,6),(4,5),(5,4),(6,3)\}|}{36} = \frac{1}{9}.$$

Also, $P(A \cap B \cap C) = \frac{1}{36} = P(A)P(B)P(C)$.
But A, B and C are not independent since

$$P(A \cap B) = \frac{1}{36} \neq P(A)P(B).$$

So, what is the correct definition for independence of more than two events?

Independence of a Collection of Events

Definition

A collection of events $A_1, ..., A_n$ are independent if

$$P(\cap_{i \in S} A_i) = \Pi_{i \in S} P(A_i)$$

for every subset S of $\{1, 2, \ldots, n\}$.

Example:

$$P(A \cap B | C \cap D) = \frac{P(A \cap B \cap C \cap D)}{C \cap D} = \frac{P(A \cap B) P(C \cap D)}{P(C \cap D)} = P(A \cap B).$$