EE126: Probability and Random Processes

Lecture 26: Course/Final Review

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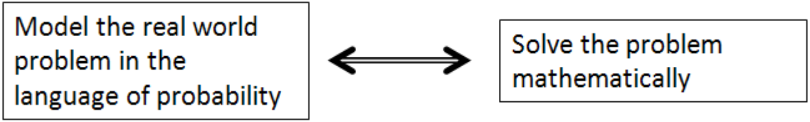
Logistics

- GSI Office Hours apply next week.
- I will be available by appointment Finals week
- Review next week on Friday
- Website will be updating by midnight tonight!
- Final is on Friday May 13 at 7PM. 277 Cory.

What did we learn?

Probability is a mathematical discipline that allows one **to reason about uncertainty**

We wanted to develop TWO equally important skills:



Understand the problem as an experiment

Combinatorics, Calculus

So What's on the Final?

- Format
  - Conceptual Section: Multiple Choice
  - Regular Problems
  - Some Extra Credit
- Material Covered
  - Everything on the first two midterms
  - Discrete Time Markov Processes: Chapter 7 (NO CONTINUOUS TIME)
  - Bayesian Estimation Chapter 8
  - Classical Stats: Only what we covered in class: Properties of Estimatators, Maximum Likelihood Estimation and Hypothesis Testing.
  - ONLY multiple choice on Chapter 9
- Cheat Sheets etc: One 8.5X11 sheet both sides + Calculator

P1: Events and Sample Spaces

Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can't be the BB card, so...
What's wrong with the reasoning that leads to $\frac{1}{2}$?
R: upturned card is Red; RB: the Red/Black card was selected.
Want $P(RB|R)$.

$$
\begin{aligned}
P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\
&= \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}(0)} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}
\end{aligned}
$$

Once you are given R: it is twice as likely that the RR card was picked.

Importance of Prior: Bayes Rule

Test for rare disease:

- If the person has the disease test is positive with prob 0.95.
- If person does not have the disease, test is negative with prob 0.95.
- A random person has the disease with prob 0.001.
- A person test +ive. What is the prob he has the disease?

Survey at a leading hospital: 80% got this wrong. Most thought 0.95!! A: has the disease; B: tests +ive

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= 0.0187$$

Independence is about Information

- A: Sum of the two dice is 6
- B: The first die is a 4
- $P(A \cap B) = P(\{4,2\}) = \frac{1}{36}$.
- $P(A) = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} = \frac{5}{36}$
- $P(B) = \frac{1}{6}$
- A and B are not independent.

- A^*: Sum of the two dice is 7
- B: The first die is a 4
- $P(A^* \cap B) = P(\{4,3\}) = \frac{1}{36}$.
- $P(A^*) = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = \frac{1}{6}$
- $P(B) = \frac{1}{6}$
- A^* and B are independent.

Why the difference??

Counting!

13 people are randomly divided into 3 groups.
Prob(two of the groups have the same number of people)=

$$\frac{\text{number of ways to get groups with two same size}}{\text{number of ways to divide into 3 groups}}$$

Number of ways to divide into 3 groups (A,B,C):3^{13}
Number of ways to get groups with two same size: i.e. k, k, 13-2k for $k = 0, 1, 2, 3, 4, 5, 6$.
For any k: Suppose that groups A&B have k. Then multiply by $\binom{3}{2}$.
Number of ways:

$$\binom{13}{k}\binom{13-k}{k}\binom{13-2k}{13-2k} = \frac{13!}{k!k!(13-k)!}$$

Multiplying by 3, summing over k, and plugging in numerator and denominator:

$$\frac{3\sum_{k=0}^{6}\frac{13!}{k!k!(13-2k)!}}{3^{13}}$$

Indicator Random Variable and Linearity of Expectation

Suppose there are n pairs of socks in the dryer and you pull out k socks at random. What is the average number of pairs will you have?
Simple Case: Let's say there are 4 socks and you pull out 3. $n = 2$ and $k = 3$.

112 121 122 221 212 211 121

Let $X_i = 1$ if pair i was pulled and zero otherwise. We want $E[\sum_{i=1}^{n} X_i]$.

$$P(X_i = 1) = \binom{k}{2}\frac{2}{2n(2n-1)} = \frac{k(k-1)}{2}\frac{1}{n(2n-1)} = \frac{k(k-1)}{2n(2n-1)}$$

So answer is $\binom{k}{2}\frac{1}{2(2n-1})$.

Words to Math

2m people form couples today. 50 years from now, the probability of any person being alive is p. Now suppose there are A people alive 50 years from now and let S be the number of couples for which both partners are still alive. Find $E[S|A=a]$.
$X_i = 1$ if the first person in couple i is alive in 50 years and $X_i = 0$ otherwise. $Y_i = 1$ if the second person in couple i is alive in 50 years and $Y_i = 0$ otherwise. Now $S = \sum_{i=1}^{m} X_i Y_i$.

$$E[S|A=a] = E[\sum_{i=1}^{m} X_i Y_i | A=a] = \sum_{i=1}^{m} E[X_i Y_i | A=a].$$

Each of the terms has the same value...

$$\begin{aligned} E[S|A=a] &= mE[X_1 Y_1 | A=a] \\ &= mE[Y_1 = 1 | X_1 = 1, A=a] P(X_1 = 1 | A=a) \\ &\quad \text{but } Y_1 \text{ is a boolean rv} \\ &= mP(Y_1 = 1 | X_1 = 1, A=a) P(X_1 = 1 | A=a) \\ &= m \frac{a-1}{2m-1} \frac{a}{2m} = \frac{a(a-1)}{2m(2m-1)} \end{aligned}$$

Why independent of p?

Power of Amnesia: Variance of a Geometric RV

$$\mathrm{var}(X) = E[X^2] - E[X]^2$$

$$\begin{aligned}
E[X^2] &= P(X=1)E[X^2|X=1] + P(X>1)E[X^2|X>1] \\
&= p + (1-p)E[(1+X)^2] \\
&= p + (1-p)E[X^2] + 1 - p + 2(1-p)\frac{1}{p} \\
&= (1-p)E[X^2] + \frac{2(1-p)}{p} \\
&= \frac{2(1-p)}{p^2}
\end{aligned}$$

$$var(X) = E[X^2] - E[X]^2 = \frac{2(1-p)}{p^2} - \frac{1}{p^2}$$

$$var(X) = \frac{1-p}{p^2}$$

Derived Distributions

$f_X(x)$ is uniform on $[a, b]$, $a \geq 0$. Find $f_Y(y)$ where $Y = X^2$.First find the CDF of Y. $y \in [a^2, b^2]$:

$$P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$F_X(\sqrt(y)) = \frac{\sqrt{y}-a}{b-a}$and $F_X(-\sqrt(y)) = 0.$

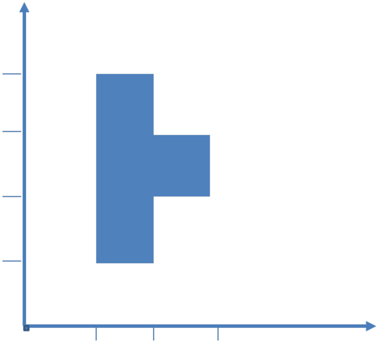
$$F_Y(y) = \begin{cases} \frac{\sqrt{y}-a}{b-a}, & 0 \leq a^2 \leq y \leq b^2 \\ 0 & \text{o.w.} \end{cases}$$

But what is $f_Y(y)$?

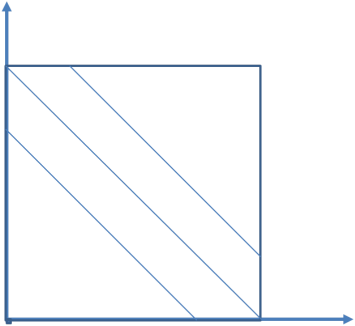
$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1}{2(b-a)\sqrt{y}}, \quad 0 \leq a^2 \leq y \leq b^2$$

Be Comfortable with Pictures

$f_{XY}(x, y) = c$ over shaded region, zero elsewhere. Find c, both marginals, $f_{Y|X}, f_{X|Y}, cov(X, Y), \rho_{X,Y}$ etc.



$Z = X + Y$



$$P(Z \leq z) = \begin{cases} \frac{1}{2}z^2, & 0 \leq z \leq 1; \\ 1 - \frac{1}{2}(2-z)^2, & 1 < z \leq 2. \end{cases}$$

$$f_Z(z) = \begin{cases} z & 0 \leq z \leq 1; \\ 2 - z, & 1 < z \leq 2. \end{cases}$$

Iterating Expectations is HELPFUL

We toss a biased coin n times. Y: prob of heads, and X: number of heads. Y is distributed uniformly over $[0,1]$. What are $E[X]$ and $var(X)$?

$$E[X|Y] = nY$$

$$\boxed{E[X] = E[E[X|Y]] = E[nY] = \frac{n}{2}}$$

$$var(E[X|Y]) = var(nY) = n^2 var(Y) = \frac{n^2}{12}$$

$$var(X|Y) = nY(1-Y)$$

$$E[var(X|Y)] = n(\frac{1}{2} - \frac{1}{3}) = \frac{n}{6}$$

$$\boxed{var(X) = \frac{n^2}{12} + \frac{n}{6}}$$

Example: Bias of a Coin Continued

Same problem: Let $X_i = 1$ if toss i is a head and $X_i = 0$ o.w.
What is $cov(X_i, X_j)$, $i \neq j$?

$$cov(X_i X_j) = E[X_i X_j] - E[X_i]E[X_j]$$

$$E[X_i] = E[E[X_i|Y]] = E[Y] = 0.5$$

$$E[X_i X_j] = E[E[X_i X_j|Y]] = E[E[X_i|Y]E[X_j|Y]] = E[Y^2] = \int_0^1 y^2 dy = \frac{1}{3}$$

$$cov(X_i X_j) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = var(Y)$$

Therefore the tosses are not independent...
Check result for $var(X)$:

$$var(X_i) = E[X_i^2] - E[X_i]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$var(X_1 + ... + X_n) = \frac{n}{4} + \frac{n(n-1)}{12} = \frac{1}{12}(3n + n^2 - n) = \frac{n^2}{12} + \frac{n}{6}$$

The CLT is KING

We want to find an estimate of p the prob a randomly chosen voter supports candidate Bob.
We ask 100 randomly sampled voters whether they support him.
$X_i = 1$ if the i^{th} voter says "yes" and $X_i = 0$ otherwise. The X_i are iid. By Chebyshev:

$$P(|M_n - p| \geq 0.1) \leq \frac{p(1-p)}{100\ (0.1)^2} \leq p(1-p)$$

X_i has bounded range $[0,1]$, and so $var(X_i) \leq \frac{1}{4}$. So:

$$\boxed{P(|M_n - p| \geq 0.1) \leq \frac{1}{4}}.$$

Now let's apply CLT.

$$P(|M_n - p| \geq 0.1) \approx 2P(M_n - p \geq 0.1)$$

Again, assuming that $var(X_i) = \frac{1}{4}$:

$$\begin{aligned} 2P(M_n - p \geq 0.1) &= 2P(\frac{M_n - p}{\sqrt{(1/4)(1/100)}}) \geq \frac{0.1}{\sqrt{(1/4)(1/100)}} = (0.1)(20)) \\ &= 2(1 - \phi(2)) = \boxed{0.046} \end{aligned}$$

CLT is much more accurate.

Bernoulli Processes Renew at each time slot

A packet arrives at a network device at the beginning of each time slot with probability p. It takes 1 slot to process. A Busy/Idle Period is a maximal string of busy/idle slots. Find the distributions of:

- Time to first busy slot: Geometric with p
- Time to first idle slot: Geometric with $1 - p$
- Length of first busy period: Geometric with $1 - p$
- Length of the first idle period: Geometric with with p

Poisson Processes and Easy if you Remember that they are Memoryless

Students arrive to vote for an election according to a Poisson Process of rate $\lambda = 30$ per hour. The voters independently vote for candidate A and B with probability $\frac{1}{2}$. Assume that the voting starts at time zero and continues indefinitely.

(a) Conditional on 300 voters arriving in the first 10 hours of voting, what is the probability that candidate A receives n of those votes in the first 4 hours of voting?The 300 votes are iid uniformly distributed on $[0, 10]$. the number of votes in $[0, 4]$ is a binomial with parameters 300 and 0.4.

(b) Find the pmf of the number of votes candidate B has received just before candidate A receives their first vote.
Let X be # B votes before first A vote. Success = candidate A vote. # votes until A gets a vote is Geometric with prob 0.5. So $X + 1$ is geometric with parameter p

(c) Define the n^{th} vote as a reversal if the n^{th} candidate votes for a different candidate than the $n - 1$st. Find the expected time between reversals.
Suppose that the reversal starts with an B vote. Then time to next A vote is time to next reversal. Since B is memoryless, expected time to this is just $\frac{1}{15}$.

(e) EXTRA CREDIT: Define a "A-to-B" reversal as a vote for candidate B that immediate follows one for candidate A. Find the probability density function of the time between two successive "A-to-B" reversals.

Lame Fishing Example

Alice commutes between two houses (A and B) via boat. She owns n fishing rods. The rods are stored at the houses. If the weather is nice at the beginning of a trip (with prob p on each trip) she grabs a rod and fishes on her trip. What is the prob that Alice wants to grab a rod but there are none at her current location? Use a n+1-state MC.

Recognize that Alice is equally likely to be at either house. Then think in terms of round trips! State j: Alice is at A and there are j rods there.

Lame Fishing Example

Birth-Death Process! $b_0 = p$, $b_i = p(1-p)$ for $1, 2, .., n-1$.
$d_i = p(1-p)$ for $i = 1, 2, ..., n$.
Local Balance Equations

$$\pi_i = \frac{p.(p(1-p))^{i-1}}{(p(1-p))^i}\pi_0 = \frac{\pi_0}{1-p}$$

for $i = 1, 2, ..., n$.
Thus

$$\pi_0 = (1 + \frac{n}{1-p})^{-1}$$

We want the probability that she is

1. At house A, in state 0 and the weather is good.This is $\pi_0 p$
2. At house B, no rods there and the weather is good.Same MC, same value of π_0, therefore $\pi_0 p$

Answer:Prob(she is at house $A)\pi_0 p$ + Prob(she is at house $B)\pi_0 p = \pi_0 p$.

Bayesian Estimation= Cool but Many Calculations

We want to Θ based on a single observation of $X = \sqrt{\Theta}W$ where W is independent of Θ has zero mean, variance 1 and known fourth moment $E[W^4]$. The mean of Θ is μ and variance σ^2. Find the Linear LMS based on X and the Linear LMS based on X^2.

$$\hat{\Theta} = E[\Theta] + \frac{cov(X,\Theta)}{\sigma_X^2}(X - E[X])$$

$E[X] = 0$ and $E[X\Theta] = E[E[X|\Theta]] = 0$. So $\boxed{\hat{\Theta} = \mu}$.
When X^2 is used we have:

$$\hat{\Theta} = E[\Theta] + \frac{cov(X^2,\Theta)}{\mathrm{var}(X^2)}(X^2 - E[X^2])$$

$E[X^2] = E[\Theta W^2] = \mu$,

$var(X^2) = var(\Theta W^2) = E[(\theta W^2)^2] - \mu^2 = (\sigma^2 + \mu^2)E[W^4] - \mu^2$

$$cov(X^2,\Theta) = E[\Theta^2 W^2] - \mu^2 = (\mu^2 + \sigma^2)(1) - \mu^2 = \sigma^2$$

$$\boxed{\hat{\Theta} = \mu + \frac{\sigma^2}{(\sigma^2 + \mu^2)E[W^4] - \mu^2}(X^2 - \mu)}$$

Last Slide

- Chance is challenging, interesting and potentially deadly
- But Chance Favors the Prepared Mind (Pasteur)
- We have learned how deal with Chance.

I am sure that you are better prepared.
Thank you and Good Luck!