EE126: Probability and Random Processes

Lecture 22: Markov Processes

Abhay Parekh

UC Berkeley

April 14, 2011

Midterm

- Turned out to be another hard exam! Low Mean!

Midterm

- Turned out to be another hard exam! Low Mean!
- Take home option:

Midterm

- Turned out to be another hard exam! Low Mean!
- Take home option:
  - Download the exam from bspace

Midterm

- Turned out to be another hard exam! Low Mean!
- Take home option:
  - Download the exam from bspace
  - Take it closed book and turn it in on Tuesday before class

Midterm

- Turned out to be another hard exam! Low Mean!
- Take home option:
  - Download the exam from bspace
  - Take it closed book and turn it in on Tuesday before class
  - Gain 0 to 25 more points:
    Delta = max(0, Take home score - In class score)
  - Final Score = In class score + Delta
  - Final Scores released on Thursday.
  - HW due on Friday instead of Wed.

Example of how to solve midterm problems

The joint pdf of two random variables, X, Y is given by $f_{X,Y}(x, y) = 1$ as long as the point (x, y) lies in a right triangle defined by the points $(0, 0), (1, 0)$, and $(0, 2)$. Find the pdf of $X + Y$.

Draw a picture.

Game Plan to the Final

- Today: Finish Markov
- Then three lectures on Estimation
- Last lecture (4/28) is a Review of the Course and for the Final
- Review sessions during the following week
- Final Date May 13, Cory 277 7-10pm

Communicating States

Accessible States: If i and j are connected by a directed path from i to j in the MC then $r_{ij}(n) > 0$ for large enough n and j is accessible from i.

A state in a MC is either Recurrent or Transient:

1. **Recurrent States:** Let $A(i)$ be the states accessible from i. Then i is recurrent if for every state $j \in A(i)$ it holds that $i \in A(j)$.
   If i is recurrent then every state in $A(i)$ must be recurrent as well. This forms a **recurrent class**.

Communicating States

Accessible States: If i and j are connected by a directed path from i to j in the MC then $r_{ij}(n) > 0$ for large enough n and j is accessible from i.

A state in a MC is either Recurrent or Transient:

1. **Recurrent States:** Let $A(i)$ be the states accessible from i. Then i is recurrent if for every state $j \in A(i)$ it holds that $i \in A(j)$.
   If i is recurrent then every state in $A(i)$ must be recurrent as well. This forms a **recurrent class**. Hotel California.

Communicating States

Accessible States: If i and j are connected by a directed path from i to j in the MC then $r_{ij}(n) > 0$ for large enough n and j is accessible from i.

A state in a MC is either Recurrent or Transient:

1. **Recurrent States:** Let $A(i)$ be the states accessible from i. Then i is recurrent if for every state $j \in A(i)$ it holds that $i \in A(j)$.
   If i is recurrent then every state in $A(i)$ must be recurrent as well. This forms a **recurrent class**. Hotel California.
2. **Transient States:** There is a state $j \in A(i)$ that cannot access i.

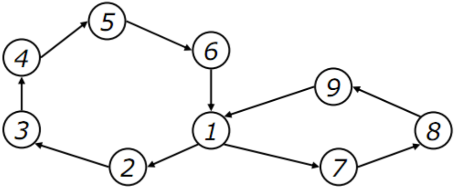
Markov Chain Decomposition

- A MC has at least one recurrent class
- If a MC has more than two recurrent classes two states from two different recurrent classes are inaccessible
- At least one recurrent state is accessible from a transient state.

Recurrent Class Periodicity

The states in the Recurrent Class of a Markov Chain are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
The period of a state i is defined by:

$$\gcd\{n : r_{ii}(n) > 0\}$$



Steady State Convergence

Any MC with a single aperiodic recurrent class must converge in the sense that

1. For each state j:

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \quad \text{for all } i$$

2. π_j are given by the system of equations:

$$\begin{aligned} \pi_j &= \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, 2, ..., m \\ 1 &= \sum_{k=1}^{m} pi_k \end{aligned}$$

3. $\pi_j = 0$ for all transient states j.
   $\pi_j > 0$ for all recurrent states j.

Local Balance Equations

Cut the MC so that the recurrent states are partitioned into two sets say A and B. The long term average number of transitions from edges that go from A to B must be equal to the long term avg of transitions from edges that go from B to A:

Local Balance Equations

Cut the MC so that the recurrent states are partitioned into two sets say A and B. The long term average number of transitions from edges that go from A to B must be equal to the long term avg of transitions from edges that go from B to A:
Example: Birth Death

$$\pi_i b_i = \pi_{i+1} d_{i+1}, \quad i = 0, 1, ..., m-1$$

$$\pi_i = \frac{b_0 b_1 ... b_{i-1}}{d_1, ..., d_i} \pi_0, \quad i = 1, 2, ..., m$$

Example: M/M/1 Queue with m buffers

Birth Death process with $b_i = \lambda\delta$, for $i = 0, 1, 2, ..., m$. $d_i = \lambda\mu$ for $i = 1, 2, ..., m$.
So $\pi_i = (\frac{\lambda}{\mu})^i \pi_0$ for $i = 1, 2, ..., m$.

Example: M/M/1 Queue with m buffers

Birth Death process with $b_i = \lambda\delta$, for $i = 0, 1, 2, ..., m$. $d_i = \lambda\mu$ for $i = 1, 2, ..., m$.
So $\pi_i = (\frac{\lambda}{\mu})^i \pi_0$ for $i = 1, 2, ..., m$.
Since $\sum_{i=0}^{m} (\frac{\lambda}{\mu})^i \pi_0 = 1$,

$$\pi_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - (\frac{\lambda}{\mu})^{m+1}}, \quad \lambda < \mu$$

M/M/1 Queue

Let $\rho = \frac{\lambda}{\mu}$. Think of it as the **utilization** of the queue. Then

$$\pi_0 = \begin{cases} \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

$$\pi_i = \begin{cases} \rho^i \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

M/M/1 Queue

Let $\rho = \frac{\lambda}{\mu}$. Think of it as the **utilization** of the queue. Then

$$\pi_0 = \begin{cases} \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

$$\pi_i = \begin{cases} \rho^i \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

Suppose m is very large. Let us look at two cases:

1. $\rho < 1$
2. $\rho \geq 1$

M/M/1 Queue: $\rho < 1$

For large m and $\rho < 1$:

$$\pi_i \approx \rho^i(1-\rho)$$

for all i. Let X: number in queue.

M/M/1 Queue: $\rho < 1$

For large m and $\rho < 1$:

$$\pi_i \approx \rho^i (1 - \rho)$$

for all i. Let X: number in queue. Then $X - 1$ is geometric with parameter $1 - \rho$.

$$E[X] =$$

M/M/1 Queue: $\rho < 1$

For large m and $\rho < 1$:

$$\pi_i \approx \rho^i(1-\rho)$$

for all i. Let X: number in queue. Then $X-1$ is geometric with parameter $1-\rho$.

$$E[X] = \frac{1}{1-\rho} - 1 = \frac{\rho}{1-\rho}.$$

Let T_i be the waiting time for arrival i.

M/M/1 Queue: $\rho < 1$

For large m and $\rho < 1$:

$$\pi_i \approx \rho^i(1-\rho)$$

for all i. Let X: number in queue. Then $X - 1$ is geometric with parameter $1 - \rho$.

$$E[X] = \frac{1}{1-\rho} - 1 = \frac{\rho}{1-\rho}.$$

Let T_i be the waiting time for arrival i.Then $E[X] = \lambda E[T]$ (Little's Law) So

$$T = \frac{1}{\mu - \lambda}.$$

Coffee Central

- Space for up to 10 people to wait comfortably.
- Want the average wait to be less than 5 mins
- Time to take order and ring up is exponential with mean 1.5 mins.

We want

$$\sum_{i=0}^{10} \pi_i \geq 0.95, \qquad T \leq 3.$$

Coffee Central

- Space for up to 10 people to wait comfortably.
- Want the average wait to be less than 5 mins
- Time to take order and ring up is exponential with mean 1.5 mins.

We want

$$\sum_{i=0}^{10} \pi_i \geq 0.95, \qquad T \leq 3.$$

$$\sum_{i=0}^{10} \pi_i = (1-\rho)\sum_{i=0}^{10} \rho^i = 1 - \rho^{11} \geq 0.95$$

So $\rho < 0.77, \Rightarrow \lambda < 0.51$.

Coffee Central

- Space for up to 10 people to wait comfortably.
- Want the average wait to be less than 5 mins
- Time to take order and ring up is exponential with mean 1.5 mins.

We want

$$\sum_{i=0}^{10} \pi_i \geq 0.95, \qquad T \leq 3.$$

$$\sum_{i=0}^{10} \pi_i = (1-\rho) \sum_{i=0}^{10} \rho^i = 1 - \rho^{11} \geq 0.95$$

So $\rho < 0.77, \Rightarrow \lambda < 0.51$.Also

$$T = \frac{1}{\frac{2}{3} - \lambda} \leq 5$$

So $\lambda \leq \frac{7}{15} < 0.47$

Coffee Central

- Space for up to 10 people to wait comfortably.
- Want the average wait to be less than 5 mins
- Time to take order and ring up is exponential with mean 1.5 mins.

We want

$$\sum_{i=0}^{10} \pi_i \geq 0.95, \qquad T \leq 3.$$

$$\sum_{i=0}^{10} \pi_i = (1-\rho) \sum_{i=0}^{10} \rho^i = 1 - \rho^{11} \geq 0.95$$

So $\rho < 0.77, \Rightarrow \lambda < 0.51$.Also

$$T = \frac{1}{\frac{2}{3} - \lambda} \leq 5$$

So $\lambda \leq \frac{7}{15} < 0.47$
So one cashier is ok so long as customers arrive at 1 every 2 mins.

Coffee Central

- Space for up to 10 people to wait comfortably.
- Want the average wait to be less than 5 mins
- Time to take order and ring up is exponential with mean 1.5 mins.

We want

$$\sum_{i=0}^{10} \pi_i \geq 0.95, \qquad T \leq 3.$$

$$\sum_{i=0}^{10} \pi_i = (1-\rho)\sum_{i=0}^{10} \rho^i = 1 - \rho^{11} \geq 0.95$$

So $\rho < 0.77, \Rightarrow \lambda < 0.51$.Also

$$T = \frac{1}{\frac{2}{3} - \lambda} \leq 5$$

So $\lambda \leq \frac{7}{15} < 0.47$
So one cashier is ok so long as customers arrive at 1 every 2 mins.Usually 2 cashiers are open. This allows roughly a customer every minute.

M/M/1 Queue: $\rho \geq 1$

For $\rho > 1$: The arrival rate is greater than the departure rate so the queue is unstable. Also, as $m \to \infty$ all the states are transient! For any finite m, however, they are recurrent.

M/M/1 Queue: $\rho \geq 1$

For $\rho > 1$: The arrival rate is greater than the departure rate so the queue is unstable. Also, as $m \to \infty$ all the states are transient! For any finite m, however, they are recurrent.
Interesting case: $\rho = 1$: All the states have the same prob of being : $\frac{1}{m+1}$. So $\pi_i \to 0$ as m increases and the queue is unstable. Turns out for ∞ states, all the states are transient except when $\lambda = \mu = 0.5$ in which case all the states are recurrent.

Random Walks on Undirected Graphs

Imagine a particle that starts at a vertex and traverses each of the outgoing edges of the vertex with equal probability. I.e. if there are $d(i)$ edges incident at i, the particle picks any one of these edges with probability $\frac{1}{d(i)}$.

Random Walks on Undirected Graphs

Imagine a particle that starts at a vertex and traverses each of the outgoing edges of the vertex with equal probability. I.e. if there are $d(i)$ edges incident at i, the particle picks any one of these edges with probability $\frac{1}{d(i)}$.

What (if any) is the stationary distribution, π_i at each vertex, i?

Random Walks on Undirected Graphs

Imagine a particle that starts at a vertex and traverses each of the outgoing edges of the vertex with equal probability. I.e. if there are $d(i)$ edges incident at i, the particle picks any one of these edges with probability $\frac{1}{d(i)}$.
What (if any) is the stationary distribution, π_i at each vertex, i?
Can G be periodic?

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.
- So the MC is aperiodic if and only every vertex is also part of an odd cycle.

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.
- So the MC is aperiodic if and only every vertex is also part of an odd cycle.
- Since the graph is connected, even one odd cycle is sufficient.

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.
- So the MC is aperiodic if and only every vertex is also part of an odd cycle.
- Since the graph is connected, even one odd cycle is sufficient.

A **bipartite graph** is one for which we divide the nodes into two classes so that all edges are interclass.

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.
- So the MC is aperiodic if and only every vertex is also part of an odd cycle.
- Since the graph is connected, even one odd cycle is sufficient.

A **bipartite graph** is one for which we divide the nodes into two classes so that all edges are interclass.
A graph is bipartite if and only if it has no odd cycles.

When is the MC aperiodic

- Every vertex is part of a cycle of length 2 since the edges are undirected.
- So the MC is aperiodic if and only every vertex is also part of an odd cycle.
- Since the graph is connected, even one odd cycle is sufficient.

A **bipartite graph** is one for which we divide the nodes into two classes so that all edges are interclass.
A graph is bipartite if and only if it has no odd cycles.

The MC is aperiodic if and only if the graph is not bipartite.

Stationary distribution

Write down the balance equations:

$$\pi_i = \sum_{j \in N(i)} \frac{\pi_j}{d(j)}$$

Stationary distribution

Write down the balance equations:

$$\pi_i = \sum_{j \in N(i)} \frac{\pi_j}{d(j)}$$

Let's guess that $\pi_i = Kd(i)$.

Stationary distribution

Write down the balance equations:

$$\pi_i = \sum_{j \in N(i)} \frac{\pi_j}{d(j)}$$

Let's guess that $\pi_i = Kd(i)$.

$$Kd(i) = K \sum_{j \in N(i)} \frac{d(j)}{d(j)} \Rightarrow d(i) = d(i)$$

Stationary distribution

Write down the balance equations:

$$\pi_i = \sum_{j \in N(i)} \frac{\pi_j}{d(j)}$$

Let's guess that $\pi_i = Kd(i)$.

$$Kd(i) = K \sum_{j \in N(i)} \frac{d(j)}{d(j)} \Rightarrow d(i) = d(i)$$

To find K we use

$$\sum_i \pi_i = 1 \Rightarrow K \sum_i d(i) = 1 \Rightarrow K = \frac{1}{\sum_i d(i)}$$

Final Result

We have shown that

$$\pi_i = \frac{d(i)}{\sum_j d(j)}$$

Final Result

We have shown that

$$\pi_i = \frac{d(i)}{\sum_j d(j)}$$

The sum $\sum_i d(i)$ counts each edge exactly 2 times. So

$$\sum_i d(i) = 2|E|$$

Final Result

We have shown that

$$\pi_i = \frac{d(i)}{\sum_j d(j)}$$

The sum $\sum_i d(i)$ counts each edge exactly 2 times. So

$$\sum_i d(i) = 2|E|$$

So we have shown that as long as the graph is not bipartite:

$$\pi_i = \frac{d(v)}{2|E|}$$

First Recurrent State: Absorbtion

Consider a MC with some transient states. Let s be a recurrent state. We want the prob that s is the first recurrent state visited when $X_0 = i$. Let this be a_i.

First Recurrent State: Absorbtion

Consider a MC with some transient states. Let s be a recurrent state. We want the prob that s is the first recurrent state visited when $X_0 = i$. Let this be a_i.

- Make each recurrent state absorbing. Then the MC has transient and absorbing states.

First Recurrent State: Absorbtion

Consider a MC with some transient states. Let s be a recurrent state. We want the prob that s is the first recurrent state visited when $X_0 = i$. Let this be a_i.

- Make each recurrent state absorbing. Then the MC has transient and absorbing states.
- Let T be the set of transient states.

$$a_i = \sum_{j=1}^{m} p_{ij} a_j, i \in T,$$

First Recurrent State: Absorbtion

Consider a MC with some transient states. Let s be a recurrent state. We want the prob that s is the first recurrent state visited when $X_0 = i$. Let this be a_i.

- Make each recurrent state absorbing. Then the MC has transient and absorbing states.
- Let T be the set of transient states.

$$a_i = \sum_{j=1}^{m} p_{ij} a_j, i \in T, \quad a_s = 1,$$

First Recurrent State: Absorbtion

Consider a MC with some transient states. Let s be a recurrent state. We want the prob that s is the first recurrent state visited when $X_0 = i$. Let this be a_i.

- Make each recurrent state absorbing. Then the MC has transient and absorbing states.
- Let T be the set of transient states.

$$a_i = \sum_{j=1}^{m} p_{ij} a_j, i \in T, \quad a_s = 1, \quad a_i = 0, \text{if } i \text{ is not } s \text{ but recurrent}$$

This system of equations always has a unique solution.

Example: Gambler's Ruin

Bob wins a dollar with probability p and loses one with prob $1-p$. He starts with with d dollars. If he has M dollars he not allowed to bet any more and he leaves with $M \geq d$ dollars. If he is bankrupt, he leaves the game with zero dollars. What is the probability that he runs out of money? The MC has states $0, 1, 2, ..., M$.
$a_0 = 1, a_M = 0$

$$a_1 = pa_0 + (1-p)a_2; a_2 = pa_1 + (1-p)a_3; ...$$

Write the equations as

$$(1-p)(a_{i-1} - a_i) = p(a_i - a_{i+1})$$

Now let $\Delta_i = a_i - a_{i+1}$ and $\rho = \frac{1-p}{p}$. Then

$$\Delta_i = \rho\Delta_{i-1} \Rightarrow \Delta_i = \rho^i\Delta_0$$

Gambler's Ruin continued

$$\sum_{i=0}^{M} \Delta_i = a_0 - a_1 + a_1 - a_2 + \cdots + a_{M-1} - a_M = a_0 - a_M = 1.$$

$$\Delta_0 = \frac{1}{1 + \rho + \rho^2 + \cdots + \rho^{M-1}}$$

Thus:

$$a_d = a_0 - (\Delta_0 + \Delta_1 + \ldots + \Delta_{d-1}) = 1 - \frac{1 + \rho + \rho^2 + \ldots + \rho^{i-1}}{1 + \rho + \rho^2 + \ldots + \rho^{M-1}}$$

$$a_d = 1 - \frac{1 - \rho^i}{1 - \rho^m}$$

Expected Time to Absorbtion

The expected times to absorbtion $\mu_1, ..., \mu_m$ are the unique solution to the equations

$$\mu_i = 0, \; i \text{ recurrent}$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j \quad i \text{ transient}$$

Return times

Mean First Passage Time and Recurrence Times

Consider a MC with a single recurrent class and let s be a particular recurrent state.

- The Mean First Passage time is the expected time for a recurrent state s to be reached from some state i:
  The mean first passage times t_i to reach s starting from i are given by

$$t_s = 0, \;\; t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, \;\; \text{for all } i \neq s$$

Return times

Mean First Passage Time and Recurrence Times

Consider a MC with a single recurrent class and let s be a particular recurrent state.

- The Mean First Passage time is the expected time for a recurrent state s to be reached from some state i:
  The mean first passage times t_i to reach s starting from i are given by

$$t_s = 0, \;\; t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, \;\; \text{for all } i \neq s$$

- The mean recurrence time is the expected time that a state s takes to return to itself.
  The mean recurrence time t_s^* of state s is given by

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j$$

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?Want mean first passage time from state 3 to state 1.

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?Want mean first passage time from state 3 to state 1.

$$\boxed{t_1 = 0}$$

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?Want mean first passage time from state 3 to state 1.

$$\boxed{t_1 = 0}$$

$$t_2 = 1 + 0.2t_2 + 0.1t_3 \Rightarrow \boxed{0.8t_2 = 0.2t_3 + 1}$$

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?Want mean first passage time from state 3 to state 1.

$$\boxed{t_1 = 0}$$

$$t_2 = 1 + 0.2t_2 + 0.1t_3 \Rightarrow \boxed{0.8t_2 = 0.2t_3 + 1}$$

$$t_3 = 1 + +0.5t_2 + 0.2t_3 \Rightarrow \boxed{0.5t_2 = 0.8t_3 - 1}$$

Example: Catching up

States: Up-to-date, Somewhat lost, Badly Lost.

$$P = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0.7 & 0.2 & 0.1 \\ 0.2 & 0.5 & 0.2 \end{pmatrix}$$

What is the Expected # weeks for Alice to be up-to-date given that Alice is badly lost?Want mean first passage time from state 3 to state 1.

$$\boxed{t_1 = 0}$$

$$t_2 = 1 + 0.2t_2 + 0.1t_3 \Rightarrow \boxed{0.8t_2 = 0.2t_3 + 1}$$

$$t_3 = 1 + +0.5t_2 + 0.2t_3 \Rightarrow \boxed{0.5t_2 = 0.8t_3 - 1}$$

Solve for t_3: $t_3 = 65/27 \approx 2.4$ weeks.