EE126: Probability and Random Processes

Lecture 21: Markov Processes

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1. Logistics

2. Review

Midterm

Will cover Sections 3.4-6.2 of the book, i.e. what we covered in Lecture from

- No questions will explicitly be on earlier material
- However, there are many dependencies!
- Same cheatsheet, calculator rules as last times.
- Read lecture notes and book. Do problems.
- Review Sessions: Sunday (4-6, Room TBA) and Monday. Stay tuned for bspace announcements.

Logistics

- Lecture notes should be updated.
- Old exams have been posted.
- HW 6 returned Tues, HW 7 today.

Communicating States

Accessible States: If i and j are connected by a directed path from i to j in the MC then $r_{ij}(n) > 0$ for large enough n and j is accessible from i.

A state in a MC is either Recurrent or Transient:

1. **Recurrent States:** Let $A(i)$ be the states accessible from i. Then i is recurrent if for every state $j \in A(i)$ it holds that $i \in A(j)$.
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2. **Transient States:** There is a state $j \in A(i)$ that cannot access i.

Markov Chain Decomposition

- A MC has at least one recurrent class
- If a MC has more than two recurrent classes two states from two different recurrent classes are inaccessible
- At least one recurrent state is accessible from a transient state.

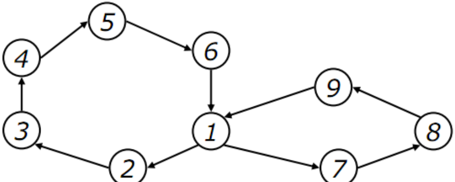
Recurrent Class Periodicity

The states in the Recurrent Class of a Markov Chain are periodic if they can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.
The period of a state i is defined by:

$$\gcd\{n : r_{ii}(n) > 0\}$$

If a recurrent class is not periodic then for any state i in the class then there must be a time n beyond which

$$r_{ii}(n) > 0$$



Steady State Convergence

Any MC with a single aperiodic recurrent class must converge in the sense that

1. For each state j:

$$\lim_{n\to\infty} r_{ij} = \pi_j, \quad \text{for all } i$$

2. π_j are given by the system of equations:

$$\begin{aligned} \pi_j &= \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, 2, ..., m \\ 1 &= \sum_{k=1}^{m} pi_k \end{aligned}$$

3. $\pi_j = 0$ for all transient states j.
$\pi_j > 0$ for all recurrent states j.

Local Balance Equations

Cut the MC so that the recurrent states are partitioned into two sets say A and B. The long term average number of transitions from edges that go from A to B must be equal to the long term avg of transitions from edges that go from B to A:

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Example: Birth Death

$$\pi_i b_i = \pi_{i+1} d_{i+1}, \quad i = 0, 1, ..., m-1$$

$$\pi_i = \frac{b_0 b_1 \ldots b_{i-1}}{d_1, ..., d_i} \pi_0, \quad i = 1, 2, ..., m$$

Example: M/M/1 Queue with m buffers

Birth Death process with $b_i = \lambda\delta$, for $i = 0, 1, 2, ..., m$. $d_i = \lambda\mu$ for $i = 1, 2, ..., m$.
So $\pi_i = (\frac{\lambda}{\mu})^i \pi_0$ for $i = 1, 2, ..., m$.

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So $\pi_i = (\frac{\lambda}{\mu})^i \pi_0$ for $i = 1, 2, ..., m$.
Since $\sum_{i=0}^{m} (\frac{\lambda}{\mu})^i \pi_0 = 1$,

$$\pi_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - (\frac{\lambda}{\mu})^{m+1}}, \quad \lambda < \mu$$

M/M/1 Queue

Let $\rho = \frac{\lambda}{\mu}$. Think of it as the **utilization** of the queue. Then

$$\pi_0 = \begin{cases} \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

$$\pi_i = \begin{cases} \rho^i \frac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1; \\ \frac{1}{m+1}, & \rho = 1. \end{cases}$$

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Suppose m is very large. Let us look at two cases:

1. $\rho < 1$
2. $\rho \geq 1$

M/M/1 Queue: $\rho < 1$

For large m and $\rho < 1$:

$$\pi_i \approx \rho^i (1 - \rho)$$

for all i. Let X: number in queue.

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$$E[X] = 1 + \frac{1}{1-\rho} = \frac{\rho}{1-\rho}.$$

Let T_i be the waiting time for arrival i.

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Let T_i be the waiting time for arrival i.Then $E[X] = \lambda E[T]$ (Little's Law) So

$$T = \frac{1}{\mu - \lambda}.$$

M/M/1 Queue: $\rho \geq 1$

For $\rho > 1$: The arrival rate is greater than the departure rate so the queue is unstable. Also, as $m \to \infty$ all the states are transient! For any finite m, however, they are recurrent.

M/M/1 Queue: $\rho \geq 1$

For $\rho > 1$: The arrival rate is greater than the departure rate so the queue is unstable. Also, as $m \to \infty$ all the states are transient! For any finite m, however, they are recurrent.
Interesting case: $\rho = 1$: All the states have the same prob of being : $\frac{1}{m+1}$. So $\pi_i \to 0$ as m increases and the queue is unstable. Turns out for ∞ states, all the states are transient except when $\lambda = \mu = 0.5$ in which case all the states are recurrent.

Head of the Line Blocking for Input Queued Switch

At each time slot, router tries to serve the head of line of each input queue. It can deliver at most 1 packet to any output port. If there is always a packet at the head of each queue and it is destined for each port with probability $\frac{1}{n}$, what is the expected throughput of the router?

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For $n = 2$ throughput loss is 25%. Gets worse with increasing n.
$< 59\%$.

Lame Fishing Example #12345

Alice commutes between two houses via boat. She owns n fishing rods. The rods are stored at the houses. If the weather is nice at the beginning of a trip (with prob p on each trip) she grabs a rod and fishes on her trip. What is the prob that Alice wants to grab a rod but there are none at her current location? (Use a n+1-state MC).

Think in terms of round trips! Call the two houses A and B. State j: Alice is at A and there are j rods there.

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$$\pi_i = \frac{p.(p(1-p))^{i-1}}{(p(1-p))^i}\pi_0 = \frac{\pi_0}{1-p}$$

for $i = 1, 2, ..., n$.
Thus

$$\pi_0 = (1 + \frac{n}{1-p})^{-1}$$

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Answer:

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Answer:Prob(she is at house $A)\pi_0 p$ + Prob(she is at house $B)\pi_0 p = \pi_0 p$.

Absorbtion

Consider a Markov Chain with multiple recurrent classes. The steady state behavior depends on which class is entered first.
For each recurrent class, make each of its states absorbing.
Then the MC has transient and absorbing states. Pick a particular absorbing state s and let a_i be the probability that $X_0 = i$ and X_n becomes equal to s for some n. Let T be the set of transient states

$$a_s = 1, \ a_i = 0, i \notin T,$$

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$$a_s = 1, \; a_i = 0, i \notin T, \; a_i = \sum_{j=1}^{m} p_{ij} a_j, i \in T$$

This system of equations always has a unique solution.

Example: Gambler's Ruin

Bob wins a dollar with probability p and loses one with prob $1 - p$. He starts with with d dollars. If he is up D dollars he not allowed to bet any more and he leaves with $D + d$ dollars. What is the probability that he runs out of money. The MC has states $0, 1, 2, ..., d + D = M$. $a_0 = 1, a_M = 1$
$a_1 = pa_0 + (1 - p)a_2; a_2 = pa_1 + (1 - p)a_3,\ a_{d+D-1} = pa_{d+D-2}$
Write the equations as

$$(1 - p)(a_{i-1} - a_i) = p(a_i - a_{i+1})$$

Now let $\Delta_i = a_i - a_{i+1}$ and $\rho = \frac{1-p}{p}$. Then

$$\Delta_i = \rho \Delta_{i-1} \Rightarrow \Delta_i = \rho^i \Delta_0$$

$$\sum_{i=0}^{M} \Delta_i = a_0 - a_1 + a_1 - a_2 + +a_{M-1} - a_M = a_1 - a_M = 1.$$

$$\Delta_0 = \frac{1}{1 + \rho + \rho^2 + \cdots + \rho^{M-1}}$$

Thus:

$$a_d = a_0 - (\Delta_0 + \Delta_1 + ... + \Delta_{d-1}) = 1 - \frac{1 + \rho + \rho^2 + ... + \rho^{i-1}}{1 + \rho + \rho^2 + ... + \rho^{M-1}} = 1 - \frac{1 - \rho^i}{1 - \rho^m}$$

Expected Time to Absorbtion

The expected times to absorbtion $\mu_1, ..., \mu_m$ are the unique solution to the equations

$$\mu_i = 0, \ \ i \text{ recurrent}$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j \quad i \text{ transient}$$

Mean First Passage Time and Recurrence Times

Consider a MC with a single recurrent class and let s be a particular recurrent state:

- The mean first passage times t_i to reach s starting from i are given by

$$t_s = 0, \ \ t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, \ \text{for all } i \neq s$$

- The mean recurrence time t_s^* of state s is given by

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j$$

Parrondo's Paradox

- Game A: Repeatedly flip a coin with bias $p_a < 0.5$. You win 1 dollar if heads, lose 1 dollar if tails. $E[W_A] < 0$.
- Game B: Two coins: Coin b_1: bias is 0.09 and Coin b_2: bias is 0.74. Start with 0 dollars. Flip coin b_1 if your winnings(losings) are a multiple of 3; else flip coin b_2. What happens here? $E[W_B]$? We will see that $E[W_B] < 0$

Parrondo's game: Flip a fair coin. It it comes up heads, play Game A; else play Game B. Surely $E[W_P] < 0$. But it isn't!
Let's see what is going on...