EE126: Probability and Random Processes

Lecture 2: Conditional Probability

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Review: The Model

Concept

1. The experiment has **base outcomes**
2. These outcomes are exhaustive and mutually exclusive
3. Each base outcome, e is assigned a non-negative number $P(e)$ called its probability
4. The probabilities summed over all of the base outcomes always equals 1

Example: Toss a fair coin twice

1. Base outcomes are HH,HT,TH,TT
2. This covers **all** the possibilities. **M.E.**
3. Each of these outcomes is equally likely
4. Assign each outcome a probability of 0.25.

The list (set) of base outcomes is called the **Sample Space**.
The rules by which probabilities are assigned are the **Axioms of Probability.**

Review: The Axioms of Probability

Any subset of the Sample Space Ω is an Event

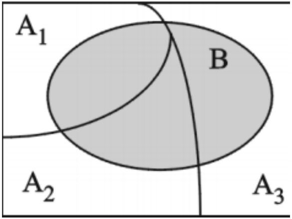
Probabilities are assigned to EVENTS

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If $A_1, A_2, \ldots,$ are mutually exclusive

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Conditional Probability

We want probability of event A given that B has occurred. Example: "Given that today is sunny, what is the probability that it will be sunny tomorrow?"



Definition

If $P(B) \neq 0$ then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

for any event A.

$$P(A \cap B) = P(A|B)P(B)$$

Tetrahedral Dice

Toss two 4-sided dice. Let the outcome represented as (X, Y). What is $P(\max(X, Y) = k | \min(X, Y) > 2)$, $k = 1, 2, 3, 4$?

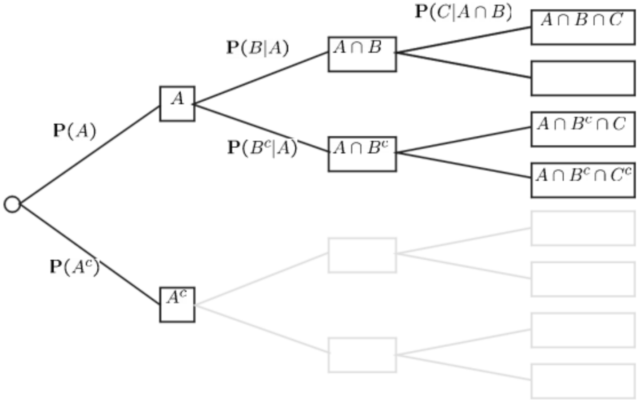
	1	2	3	4
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
3	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
4	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

$\Rightarrow$

	1	2	3	4
1				
2				
3			$\frac{1}{4}$	$\frac{1}{4}$
4			$\frac{1}{4}$	$\frac{1}{4}$

$P(\max(X, Y) = 1 | \min(X, Y) > 2) =$
$P(\max(X, Y) = 3 | \min(X, Y) > 2) =$
$P(\max(X, Y) = 4 | \min(X, Y) > 2) =$

"Tree" Interpretation



- The leaves are mutually exclusive!
- Multiplication Rule: For example

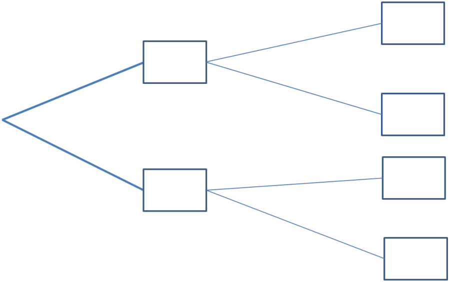
$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B).$$

Radar Example

A radar correctly detects an airplane if it is present with prob 0.99. However, it may also err in claiming an airplane is present when it is not with prob 0.1. The prob of an airplane being present is 0.05. What is the prob that

- an airplane is present but not detected?
- an airplane is not present but detected?

A: airplane present; D: airplane detected.
$P(D|A) = 0.99$; $P(D|A^c) = 0.1$; $P(A) = 0.05$
$P(A \cap D^c) = ?$, $P(A^c \cap D) = ?$



Multiplication Rule: Algebra

We know that $P(A \cap B) = P(A|B)P(B)$.

$$P(A \cap B \cap C) = P(A)\frac{P(A \cap B)}{P(A)}\frac{P(A \cap B \cap C)}{P(A \cap B)}$$

Thus

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B).$$

(We've already seen this in the examples.) In general:

Definition

Assuming $A_1, ..., A_n$ have positive probability:

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)...P(A_n|\bigcap_{i=1}^{n-1} A_i)$$

Card Drawing

Draw 3 cards at random from a 52-card deck. Find P(no hearts drawn).
Think of the draws as sequential with no replacement. Then A_i: i^{th} draw is not a heart.

$$P(\bigcap_{i=1}^{3} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

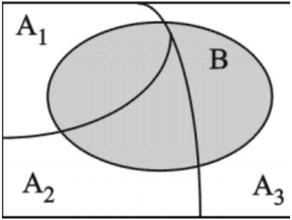
$P(A_1) = \frac{39}{52}$; $P(A_2|A_1) = \frac{38}{51}$; $P(A_3|A_1 \cap A_2) = \frac{37}{50}$.
Thus

$$P(\bigcap_{i=1}^{3} A_i) = \frac{39}{52}\,\frac{38}{51}\,\frac{37}{50}.$$

Total Probability

$A_1, ..., A_N$ are mutually exclusive, non-empty events and that $\sum_{i=1}^{N} P(A_i) = 1$.

For any event B: Each of the base outcomes that comprise B must be in exactly one of $A_1, A_2, ..., A_N$. Suppose we can calculate $P(B|A_i)$ for all i. Then we can find $P(B)$.



Total Probability Theorem

$$P(B) = P(A_1 \cap B) + \ldots + P(A_N \cap B)$$
$$= P(A_1)P(B|A_1) + \ldots + P(A_N)P(B|A_N)$$

Bayes Rule

$A_1, \ldots, A_N$ partition Ω. We know the $P(A_i)$. Also, we know $P(B|A_i)$ for all i. Want $P(A_i|B)$

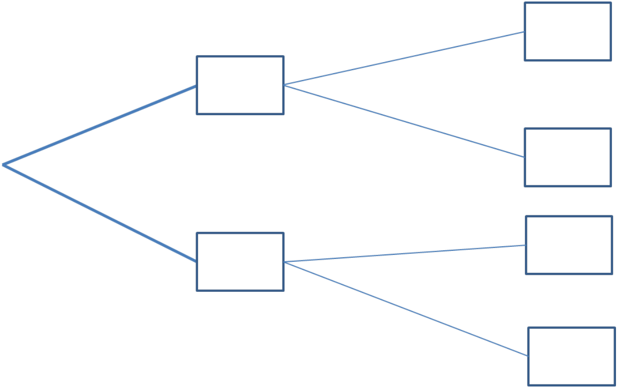
Example: An Xray has a shade (event B) which might be due to a malignant tumor (A_1), benign tumor (A_2), or other (A_3). We know $P(B|A_i)$, but what we care about is $P(A_i|B)$.

Bayes Rule

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_j^N P(A_j)P(B|A_j)}$$

Truth-telling Example

People on an island tell the truth with prob $\frac{1}{3}$. You ask one of them a question. They answer it. Then you ask another inhabitant if that answer was correct. This person says "Yes". What is the probability that the first person's answer was correct?
T the first person is truthful. Y**:** second person says "Yes". We want to calculate $P(T|Y)$.



False Positive Puzzle

Test for rare disease:

- If the person has the disease test is positive with prob 0.95.
- If person does not have the disease, test is negative with prob 0.95.
- A random person has the disease with prob 0.001.
- A person test +ive. What is the prob he has the disease?

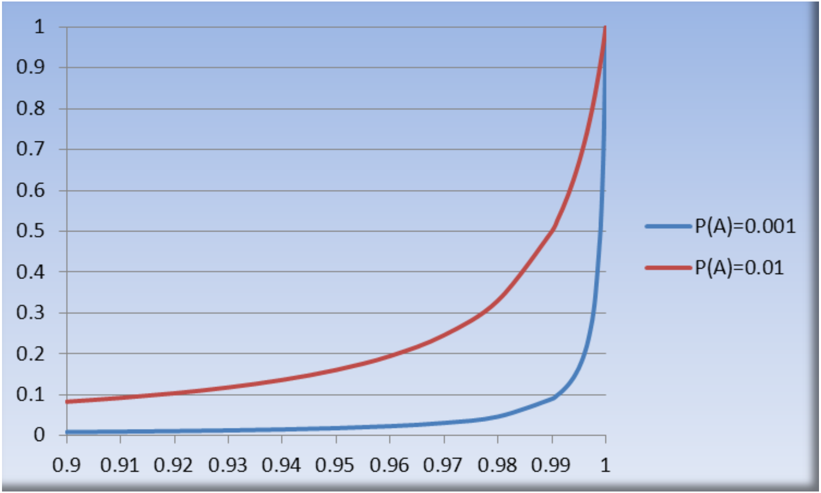
A: has the disease; B: tests +ive

$$P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)}$$

$$= 0.0187$$

Survey at a leading hospital: 80% got this wrong. Most thought $P(A|B) = 0.95$!!

Qualify before testing!



Three Card Problem

Three cards: Red/Red, Red/Black, Black/Black.
Pick one at random and place on the table. The upturned side is a Red. What is the probability that the other side is Black?
Can't be the BB card, so...
R: upturned card is Red; RB: the Red/Black card was selected.
Want $P(RB|R)$.

$$
\begin{aligned}
P(RB|R) &= \frac{P(RB \cap R)}{P(R)} \\
&= \frac{\frac{1}{3}\frac{1}{2}}{\frac{1}{3}(1) + \frac{1}{3}\frac{1}{2} + \frac{1}{3}(0)} \\
&= \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}
\end{aligned}
$$

What's wrong with the reasoning that leads to $\frac{1}{2}$?
Once you are given R: it is twice as likely that the RR card was picked.