EE126: Probability and Random Processes

Lecture 19: Poisson Process

Abhay Parekh

UC Berkeley

March 31, 2011

1. Logistics
2. Review
3. Poisson Processes

Logistics

Poisson Process

A continuous version of the Bernoulli Process, i.e. time is not in slots but defined on the non-negative reals.

1. The number of arrivals in time t is a Poisson rv with mean λt:

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

2. The interarrival times are iid exponential rvs with parameter λ.
3. "First Principles" definition: Time homogeneous and non-overlapping intervals independent. For small intervals of size δ, prob of an arrival in that interval is approx $\lambda\delta$; prob of one arrival is approx $1 - \lambda\delta$ (so prob of > 1 arrival is approx 0).

Depending on the situation, use version 1,2 or 3 to your benefit.

Example: Fishing

Bob catches fish according to a Poisson Process with rate $\lambda = 0.6$ per hour. If catches as at least one in the first two hours he quits. Else he continues until he has caught the first fish.

- Prob Bob fishes for > 2 hours: The is happens if Bob doesn't catch anything for the first 2 hours. Poisson RV: $P(N(2) = 0) = e^{-1.2}$
- Prob Bob fishes for time in $[2, 5]$ hours: This happens when Bob does not catch anything in $[0, 2]$ and catches a fish in $[2, 5]$. So no arrivals in $[0, 2]$ and at least one arrival in $[2, 5]$. Both events are independent. Prob of first event is $e^{-1.2}$. Prob second event is Prob(inter-arrival time ¡ 3)=$(1 - e^{-1.8})$. So $e^{-1.2}(1 - e^{-1.8})$.

Example: Fishing

Bob catches fish according to a Poisson Process with rate $\lambda = 0.6$ per hour. If catches as at least one in the first two hours he quits. Else he continues until he has caught the first fish.

- Prob Bob catches at least 2 fish: This can only happen if he catches at least two fishes in the first two hours. So 1-P(catches 1 fish in 2 hours) - P(catches 0 fish in 2 hours). $1 - P(N(2) = 1) - P(N(2) = 0) = 1 - e^{-1.2} - 1.2e^{-1.2}$.
- Expected number of fish caught: Total fish caught = Fish Caught in [0,2] + Fish Caught [2,]. Linearity of execrations:$1.2 + e^{-1.2}$
- Expected total fishing time given it is > 4 hours: $4 +$ E[time to catch a fish] $=$ 4+E[interarrival time]$= 4 + \frac{1}{.6} = 5\frac{2}{3}$ hours.

Merging and Splitting Poisson Processes

1. If you merge K independent PPs with rates $\lambda_1, ..., \lambda_k$ you get one PP of rate $\lambda = \sum_i \lambda_i$.
2. If you look at a random arrival in the merged PP the prob it belongs to PP i is $\frac{\lambda_i}{\lambda}$.
3. When you split a Poisson process of rate λ into k by routing each arrival independently to i with prob p_i, each of the k PPs are independent and the i^{th} has rate $p_i\lambda$.

Merging and splitting are very helpful in solving problems with "Competing" exponentials.

Example: Light Bulbs

K identical light bulbs with lifetimes iid exponential with parameter λ. What is the expected value and variance of the time until the last one burns out?
Think of a failure as an arrival of a Poisson Process. When k bulbs are working the the next failure "arrives" according to a Poisson Process with rate $k\lambda$. Initially $k = K$.
E[first failure when k bulbs working] $= \frac{1}{k\lambda}$

After a failure you have one less bulb, i.e. one less PP:E[time until K fail]$=\sum_{i=0}^{K-1} E$[1st failure with $K-i$ bulbs]

Expected value $= \sum_{i=1}^{K} \frac{1}{i\lambda}$.
Similarly variance $= \sum_{i=1}^{K} \frac{1}{i^2\lambda^2}$.

Example: Spam Folders

Valid mail and Spam mail both arrive to your inbox according to independent Poisson Processes of $\lambda_v = 4$ (4 per hour) and $\lambda_s = 2$ respectively.
Your spam filter tags a spam email correctly with prob $p = 0.8$; a valid email as spam with prob $q = 0.1$

1. Prob an email in your inbox is spam
2. Prob an email in your spam filter is valid
3. How often should you check your spam folder if you'd like to see one new valid email each time you check?

Competing Poisson Processes

Two independent Poisson Processes with rates λ_1, λ_2. Let Y_k^i be the time of the k^{th} arrival in process $i = 1, 2$.
Find $P(Y_n^1 < Y_m^2)$, the prob the nth arrival in 1 is before the mth arrival in 2.
Merge the 2 processes to get one with rate $\lambda_1 + \lambda_2$. Probability that an arrival is from 1:

$$p_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

We need n routed to 1 before m routed to 2
We need at least n "successes" before $n + m - 1$ "trials"(midterm problem!):

$$P(Y_n^1 < Y_m^2) = \sum_{k=0}^{m-1} \binom{n+m-1}{n+k} p_1^{n+k} (1 - p_1)^{m-k-1}$$

Tasks on Processors

There n tasks on n processors. Two stages to each task. Time for each stage is iid exponential with parameter 1.
Find: Prob k half done tasks when the first task is done.
There are exactly n iid competing exponentials at all times. Merge them to get a Poisson Process of rate n. An arrival means that a **stage** has been completed Now split the merged process n ways with prob $\frac{1}{n}$. An Each arrival in the merged process is tagged i with prob $\frac{1}{n}$ for $i = 1, 2, .., n$.
We want k distinct tags after we see a duplicate.
So: k+1 successive tags and then a duplicate.
In a merged process prob a given arrival came from i is $\frac{\lambda_i}{\lambda}$. So:

$$1 \; \frac{n-1}{n} \; \frac{n-2}{n} \ldots \frac{n-k}{n} \; \frac{k+1}{n}$$

This is

$$\frac{\binom{n}{k+1}(k+1)!}{n^k} \frac{k+1}{n}$$

Conditional Distribution of arrival times

Given that $N(t) = n$, what is the joint distribution of the n arrival times $Y_1, Y_2, ..., Y_n$? Let us call this distribution $f(t_1, t_2, \ldots, t_n)$:
(Try with $n = 1$ and Bernoulli....) (Try with $n = 1$ and Poisson)
For $0 < t_1 < t_2 < ... < t_n < t$ and $h_1, h_2, ..., h_n$:

$$\lim_{h_1 \ldots h_n \to 0} \frac{P(t_i \le Y_i \le t_i + h_i,\ \ i = 1, \ldots n | N(t) = n)}{h_1 h_2 \ldots h_n} = f(t_1, .., t_n)$$

We need exactly one arrival in $[t_i, t_i + h_i]$ $(i = 1, 2, .., n)$. Pick h_i small enough so that these intervals are non-overlapping. Also, no arrivals anywhere else in $[0, t]$. Thus:

$$\begin{aligned} P(t_i \le Y_i \le t_i + h_i | N(t) = n) &= \frac{\lambda h_1 e^{-\lambda h_1} \ldots \lambda h_n e^{-\lambda h_n} e^{-\lambda(t - h_1 - h_2 - \ldots h_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\ &= \frac{n!}{t^n} h_1 h_2 \cdots h_n \end{aligned}$$

$$f(t_1, ..., t_n) = \frac{n!}{t^n}.$$

Conditional Distribution of arrival times

Given that $N(t) = n$, the joint distribution of the n arrival times $Y_1, Y_2, ..., Y_n$ is

$$f(t_1, ..., t_n) = \frac{n!}{t^n}.$$

Interpretation: Given n iid random variables $X_1, .., X_n$, order them so that

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}$$

Then for $x_1 < x_2 < ... < x_n$

$$f_{X_{(1)},X_{(2)},\cdots,X_{(n)}}(x_1, ..., x_n) = n! \prod_{i=1}^{n} f_{X_i}(x_i)$$

If $X_1, ..X_n$ are iid uniform over $[0, t]$:

$$f_{X_{(1)},X_{(2)},\cdots,X_{(n)}}(x_1, ..., x_n) = \frac{n!}{t^n}$$

So given that n arrivals have occurred, $Y_1, .., Y_n$ considered as unordered random variables are iid uniform.

Customers in a bank

Two kinds of customers (A and B) enter a bank. Both arrival processes are Poisson ($\lambda_A = 3$ minutes and $\lambda_B = 4$ minutes). Type A customers are always depart in 1 min and Type 2 customers always depart in time geometrically distributed with mean $=2$.

1. N(t) customers arrive in t minutes. How is N(t) distributed?
2. 10 customers arrived in 10 minutes. Prob that 3 of type A?:
3. At $t = 0$ there are no customers in the bank.
4. At $t = 1$ exactly 1 type B job. Find mean and variance of the additional time time it must wait until it departs.

Customers in a bank

Two kinds of customers (A and B) enter a bank. Both arrival processes are Poisson ($\lambda_A = 3$ and $\lambda_B = 4$).
Type A customers are always served in 1 min and Type 2 customers are served in time geometrically distributed with mean =2.

1. N(t) customers arrive in t time units. How is N(t) distributed? Poisson with parameter 7t.
2. 10 customers arrived in 10 mins. Prob that 3 of type A?: $\binom{10}{3}(3/7)^3(4/6)^7$.
3. At $t = 0$ there are no customers in the bank. Prob k type B customers arrive before the first type A customer: $(4/7)^k(3/7)$

Customers in a bank

Two kinds of customers (A and B) enter a bank. Both arrival processes are Poisson ($\lambda_A = 3$ and $\lambda_B = 4$).
Type A customers are always served in 1 min and Type 2 customers are served in time geometrically distributed with mean =2.

(4) At $t = 1$ exactly 1 type B job. Find the mean and variance of the additional time time it must wait until it departs.

Suppose the job arrived at time $-K + X$, $-K$ integer and $X \in (0, 1)$. Since type B waiting time is memoryless the time left from $1 + X$ to its departure is geometric. Let us call this time L.
Total waiting time is distributed as $X + L$.
Given an arrival occurred in $(-K - 1, -K)$, arrival is uniformly distributed in that interval. So X is uniform in $(0, 1)$. X and L are independent:
$E[T] = 2.5$, $var[T] = 1/3$. What is the distribution of $X + L$?