EE126: Probability and Random Processes

Lecture 18: Poisson Process

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Bernoulli Process

An arrival process comprised of a sequence of coin flips: $X_1, X_2, \dots$ where the X_i are iid Bernoulli rv's with probability of success p.

1. Let $N(t)$ be the number of arrivals in t time slots. Then, $N(t) = \sum_i X_i$ is Binomial.
2. Let Y_k be the time of the k^th arrival.
   Then Y_1 is the time to the first head i.e. $Y_1 \sim$ Geometric with parameter p.
3. $T_1 = Y_1$ is the time to the first arrival. $T_2 = Y_2 - Y_1$ is the time between the first and second arrivals. Since coin flipping is memoryless, $Y_2 - Y_1$ is also geometric and independent of Y_1.

Definition 2

A sequence of iid geometric random variables with parameter p.

Distribution of the Time of the k^{th} Arrival

Y_k is the sum of the first k interarrival times.
The interarrival times are geometric rvs.
(Geometric Mean: $1/p$; Var: $(1-p)/p^2$.)

$$\boxed{E[Y_k] = \qquad\qquad}$$

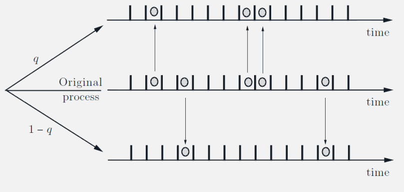
$$\boxed{var(Y_k) = \qquad\qquad\qquad}$$

In the first $t-1$ time units we have to had exactly $k-1$ arrivals.
The t^{th} time unit must register an arrival.

$$\boxed{P(Y_k = t) = \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} p = \binom{t-1}{k-1} p^k (1-p)^{t-k}}$$

Called Pascal Distribution of order k.

Splitting Bernoulli Processes

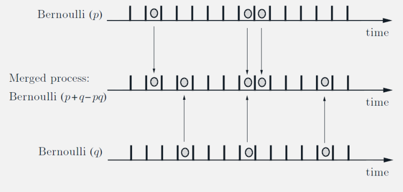


For each arrival, route it to A with prob q and to B with prob $1-q$. Routing decisions are independent.
A_i: arrival to A at time slot i, and B_i: arrival to B at slot i.

$$P(A_i) = \begin{cases} 1, & \text{if } X_i = 1 \text{ and route to } A; \\ 0, & \text{otherwise.} \end{cases}$$

So, A_i is Bernoulli with prob pq and B_i is Bernoulli with probability $p(1-q)$.

Merging Bernoulli Processes



A and B are independent Bernoulli Processes, with prob p_A and p_B respectively. Construct the arrival process C as follows:
$C_i = 1$ if $A_i = 1$ or $B_i = 1$ (includes if both have arrivals) and $C_i = 0$ otherwise.
Then $P(C_i = 0) = (1 - p_A)(1 - p_B)$ and
$P(C_i = 1) = p_C = 1 - (1 - p_A - p_B + p_A p_B) = p_A + p_B - p_A p_B$.
Thus C is Bernoulli Process with probability p_C.

Poisson Process

A continuous version of the Bernoulli Process, i.e. time is not in slots but defined on the non-negative reals.
Immediate issue...how does p change as the slot size goes to zero?
Slot size is δ: Fix $\lambda > 0$ and set $p(\delta) = \lambda\delta$ for $\lambda > 0$.

- Prob one arrival in a slot $p(\delta) = \lambda\delta$
- Prob no arrival in a slot $1 - \lambda\delta$
- Expected Number of arrivals in in time τ: There are $n = \frac{\tau}{\delta}$ slots so number of arrivals in τ is Binomial (n,p).

$$np = \lambda\tau$$

Now what happens as we let $\delta \to 0$ (but keep λ fixed)?
Let $N(t)$ the number of arrivals in t time units. $P(N(t) = k)$ converges from a Binomial to a Poisson distribution.

Review: Poisson Random Variable

Counting Successes with a Binomial rv can be computationally difficult:

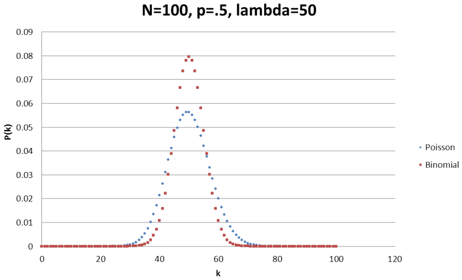
1. Count misprints in a book. N is huge and p is very small.
2. What is the probability that exactly k out 500 randomly chosen people will share a birthday on New Year's Day?
3. A factory produces defective screws with prob 0.015. What is the prob that a box of 100 screws has k defective items?

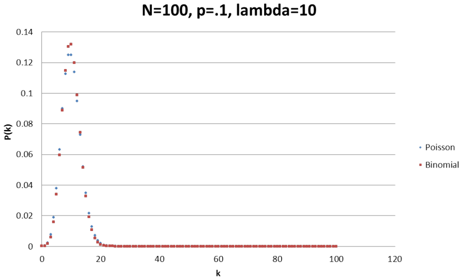
When N is large and p is small an excellent approximation is:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

where $\lambda = Np > 0$.

Approximation of Binomial by Poisson





N=100,p=.01,lambda=1

	Poisson	Binomial
0	0.3678794	0.366032
1	0.3678794	0.36973
2	0.1839397	0.184865
3	0.0613132	0.060999
4	0.0153283	0.014942
5	0.0030657	0.002898
6	0.0005109	0.000463
7	7.299E-05	6.29E-05
8	9.124E-06	7.38E-06
9	1.014E-06	7.62E-07
10	1.014E-07	7.01E-08

Why?

Fact:$e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$.
Start with Binomial...set $\lambda = np \Rightarrow p = \frac{\lambda}{n}$

$$\begin{aligned} p_X(k) &= \frac{n!}{(n-k)!k!}p^k(1-p)^{n-k} \\ &= \frac{n(n-1)\ldots(n-k+1)}{n^k}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^{n-k} \\ &= \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-k+1}{n}\frac{\lambda^k}{k!}(1-\frac{\lambda}{n})^{-k}(1-\frac{\lambda}{n})^n \end{aligned}$$

As $n \to \infty$, $p_X(k) \to e^{-\lambda}\frac{\lambda}{k!}$ when $k << n$.

Poison Process: Formal Definition

Let $N(t)$ be the number of arrivals in $[0, t]$ and let $P_k(t)$ be the probability that $N(t) = k$.

1. Time Homogeneity: $P_k(\tau)$ is the same for all intervals of length τ
2. Independence: The number of arrivals in any interval is independent of all arrival events outside that interval.
3. Small Interval Probabilities:

$$\begin{aligned} P_0(\tau) &= 1 - \lambda\tau + o_0(\tau) \\ P_1(\tau) &= \lambda\tau + o_1(\tau) \\ P_k(\tau) &= o_k(\tau) \quad \text{for } k = 2, 3, ..., \end{aligned}$$

Note:

$$\lim_{\tau \to 0} \frac{o_k(\tau)}{\tau} = 0, \quad k = 0, 1, 2,$$

Occurrence

1. The number of goals in (90 minutes of) a soccer match
2. A renewal process is an arrival process in which the interarrival times are iid but not necessarily exponential. Lots of small independent renewal processes "add up" to a Poisson Process.
   - Phone calls originating in a city
   - The arrival of "customers" in a queue.
   - Car accidents in a city
   - Particle emissions from radioactive material
   - The number of raindrops falling over an area

Poisson Process

Let the number of arrivals of a Poisson process with rate λ in an interval of length t be $N(t)$. Then

$$P(N(t) = k) = P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

for $k = 0, 1, ...$ Also:

$$E[N(t)] = var(N(t)) = \lambda t$$

Interarrival times

Starting from time τ let $\tau + T$ be the time of the next arrival.

$$\begin{aligned} P(T \leq t) &= P(N(t) > 0) = 1 - P(N(t) = 0) \\ &= 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t} \\ &= 1 - e^{-\lambda t} \end{aligned}$$

This is the CDF of an exponential r.v.

T_k: k^{th} inter-arrival time. Then the T_k are all identically distributed exponential rvs.

They describe non-overlapping intervals $\Rightarrow$ they are independent.

Given a sequence of iid exponential random variables $T_1, T_2 ...$, we can construct a Poisson process:

$$Y_1 = T_1, \quad Y_k = Y_k - Y_{k-1}, \; k = 2, 3, ...$$

where Y_k is the time of the k^{th} arrival.

Thus a Poisson Process is a process where the interarrival times are iid exponential random variables.

Time to k^{th} arrival

Y_k is the time the k^{th} arrival.

$$f_{Y_k}(y)\delta \approx P(k^{th} \text{ arrival in } [y, y+\delta])$$

- The k^th arrival occurs in the interval with prob $\approx \lambda\delta$
- The other $k-1$ arrivals occurred before y. I.e. with probability:
$$\frac{(\lambda y)^{k-1}}{(k-1)!}e^{-\lambda y}.$$
- The two events are independent
$$f_{Y_k}(y)\delta \approx \lambda\delta\frac{(\lambda y)^{k-1}}{(k-1)!}e^{-\lambda y}.$$

$$\boxed{f_{Y_k}(y) = \frac{\lambda^k y^{k-1}}{(k-1)!}e^{-\lambda y}.}$$

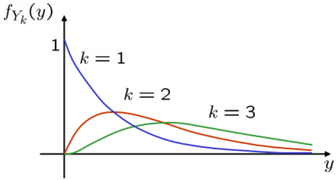
This is the Erlang PDF of order k.

Erlang Distribution

The distribution of Y_k, the k^{th} arrival time is given by

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1}}{(k-1)!} e^{-\lambda y}.$$

The k-order Erlang is the sum of k iid exponentials.



Video Game

Alice is playing a game in which she must move her player from between two points. Machine gun blasts that annihilate everything between A and B arrive according to a Poisson process with rate λ. Alice decides to move her player when no machine gun blast has arrived for τ seconds. Let N be the number of blasts she sees before moving her player.

1. $E[N]$: If $N = n$, there are n interarrivals less than τ and then one greater than τ. The interarrival times are independent and exponentially distributed with λ. So $N + 1$ is geometric with parameter $e^{-\lambda\tau}$. So

$$E[N] + 1 = e^{\lambda\tau} \Rightarrow E[N] = e^{\lambda\tau} - 1.$$

2. It takes τ_1 seconds to move the player. Find P(player is annihilated): At the time she decides to move the player, residual time of the interarrival time is exponentially distributed so $1 - e^{-\lambda\tau_1}$.

Example: Fishing

Bob catches fish according to a Poisson Process with rate $\lambda = 0.6$ per hour. If catches as at least one in the first two hours he quits. Else he continues until he has caught the first fish.

1. Prob Bob fishes for > 2 hours: $e^{-1.2}$
2. Prob Bob fishes for time in $[2, 5]$ hours: $e^{-1.2}(1 - e^{-1.8})$.
3. Prob Bob catches at least 2 fish: $1 - e^{-1.2} - 1.2e^{-1.2}$.
4. Expected number of fish caught: $1.2 + e^{-1.2}$
5. Expected total fishing time given it is > 4 hours: $4 + \frac{1}{.6} = 5\frac{2}{3}$ hours.

Merging of two Independent Poisson Processes

- Two processes, A and B with parameters λ_A and λ_B respectively.
- The merged process, C, has an arrival whenever either A or B has an arrival. It is clearly Time Homogeneous and Independent.
- In a small interval of length δ:
  1. P(0 arrivals in C)
     $\approx (1-\lambda_A\delta)(1-\lambda_B\delta) = 1-\lambda_A\delta-\lambda_B\delta-\lambda_A\lambda_B\delta^2$
     P(0 arrivals in C)$\approx 1-(\lambda_A+\lambda_B)\delta$
  2. P(1 arrival in C)
     $\approx \lambda_A\delta(1-\lambda_B\delta)+\lambda_B\delta(1-\lambda_A\delta) = (\lambda_A+\lambda_B)\delta.$

Process C is Poisson with parameter $\lambda_A+\lambda_B$.

Merging of two Independent Poisson Processes

- Two processes, A and B with parameters λ_A and λ_B respectively.
- The merged process, C, is Poisson with parameter $\lambda_A + \lambda_B$.
- Suppose a single arrival occurs in a small interval of length δ:
  1. P(1 type A arrival in $[0, \delta]$|1 arrival in $[0, \delta]$):
  $$\approx \frac{\lambda_A \delta}{(\lambda_A + \lambda_B)\delta}$$
  2. Since C is time homogeneous, this is true of any arrival in C:

The probability that a particular arrival is a type A arrival is

$$\frac{\lambda_A}{(\lambda_A + \lambda_B)}$$

Splitting of Poisson Processes

Same idea as with Bernouli: Each arrival of a PP with rate λ is routed "up" with probability p and "down" with prob $1-p$.

- Upper is Poisson with rate λp
- Lower is Poisson with rate $\lambda(1-p)$

Are two processes independent? Yes!
Let $N_u(t)$: number of arrivals to the upper process;
$N_l(t)$: number of arrivals to the lower process

$$
\begin{aligned}
P(N_u(t)=m, N_l(t)=n) &= P(N_u(t)=m, N_l(t)=n|P(N(t)=m+n)P(N(t)= \\
&= \binom{m+n}{m} p^m(1-p)^n \frac{(\lambda t)^{m+n}}{(m+n)!} e^{-\lambda t} \\
&= \frac{(m+n)!}{m!n!} p^m(1-p)^n \frac{(\lambda t)^{m+n}}{(m+n)!} e^{-\lambda t} \\
&= \frac{(\lambda t)^m}{(m)!} e^{-\lambda p t} \frac{(\lambda t)^n}{(n)!} e^{-\lambda(1-p)t}
\end{aligned}
$$

Tasks on Processors

There n tasks on n processors. Two stages to each stage. Time for each stage is iid exponential with parameter 1.
Find: Prob k half done tasks when the first task is done.
There are exactly n iid competing exponentials at all times. Merge them to get a Poisson Process of rate n. Each arrival in the merged process is tagged i with prob $\frac{1}{n}$ for $i = 1, 2, .., n$.
Prob $k+1$ successive distinct tags and $k+2^{nd}$ duplicate tag.

$$1 \; \frac{n-1}{n} \; \frac{n-2}{n} ... \frac{n-k}{n} \; \frac{k+1}{n}$$

This is

$$\frac{\binom{n}{k+1}(k+1)!}{n^k} \frac{k+1}{n}$$