EE126: Probability and Random Processes

Lecture 17: Bernouli Process

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1. Review

2. Stochastic Processes

3. Bernoulli Process

Central Limit Theorem

Variance and mean of S_n grow unboundedly with n:
$E[S_n] = n\mu$, $var(S_n) = n\sigma^2 \Rightarrow \sigma_{S_n} = \sqrt{n}\sigma$. So why not look a the normalized version of S_n?

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$$S_n \to N(n\mu, n\sigma^2) \qquad M_n \to N(\mu, \frac{\sigma^2}{n}).$$

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De Moivre-Laplace Approximation to Binomial

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$$P(k \leq S_n \leq l) \approx \phi(\frac{l + \frac{1}{2} - np}{\sqrt{np(1-p)}}) - \phi(\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}})$$

Proof of the CLT

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$X_1, ... X_n$ are iid with mean 0 and variance σ^2. Let $M_X(s)$ the transform of each of the X_i. $\hat{S}_n = \frac{\sum_i X_i}{\sqrt{\sigma n}}$ has mean 0 and var 1.

1. Since the X_i are independent:

$$M_{\hat{S}_n}(s) = \Pi_{i=1}^n E[e^{\frac{X_i}{\sqrt{\sigma^2 n}}}] = (M_X(\frac{s}{\sigma\sqrt{n}}))^n$$

2. Find Taylor Expansion of $M_X(s)$ around $s = 0$.

$$M_X(s) = 1 + \frac{\sigma^2}{2}s^2 + o(s^2).$$

$$M_{\hat{S}_n}(s) = (M_X(\frac{s}{\sigma\sqrt{n}}))^n = (1 + \frac{s^2}{2n} + o(\frac{s^2}{\sigma^2 n}))^n.$$

3. As we take limits, only the second term remains!

$$\lim_{n\to\infty} \ln M_{\hat{S}_n}(s) = \lim_{n\to\infty} \ln(1 + \frac{s^2}{2n} + o(\frac{s^2}{\sigma^2 n}))^n$$

But this goes to $\to \frac{s^2}{2}, \quad \text{as } n \to \infty$. This is the log of the transform for a Standard Normal.

4. Thus for any x, $F_X(x) \to$CDF of a Standard Normal. I.e., $\hat{S}_n$ **converges in distribution** to $N(0, 1)$.

Strong Law of Large Numbers

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We say that M_n **converges to** μ **almost surely**.

What's the difference between the Weak and Strong Laws?

If $X_1, ..., X_n$ are iid random variables with mean μ and $M_n = \frac{1}{n}\sum_i X_i$:

Weak Law of Large Numbers

For every $\epsilon > 0$:

$$\lim_{n\to\infty} P(|M_n - \mu| \geq \epsilon) = 0$$

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$$P(\lim_{n\to\infty} M_n - \mu = 0) = 1.$$

Point of difference: For some $\epsilon > 0$, how often is $|M_n - \mu| > \epsilon$?

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2. Strong Law: Has to be a finite number of times for $\lim_{n\to\infty} M_n - \mu$ to be zero.

Example

Suppose time is in discrete units, i.e., $1, 2, ...,$ and $Y_n = 1$ if there is an arrival at time n and $Y_n = 0$ otherwise.
Define $I_k = \{2^k, 2^k + 1, ..., 2^{k+1} - 1\}$ so that:

$$I_1 = \{1\},\ I_2 = \{2, 3\},\ I_3 = \{4, 5, 6, 7\} \quad \text{etc.}$$

Suppose that during each interval I_k we have exactly one arrival, and that arrival is equally likely to be at any time in that interval.

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Then $P(Y_n = 1) = \frac{1}{2^k}$ if $n \in I_k$.

$$\lim_{n\to\infty} P(Y_n = 1) = \lim_{k\to\infty} \frac{1}{2^k} = 0.$$

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Thus, Y_n converges to 0 in probability.
However, $P(\lim_{n\to\infty} Y_n = 0) \neq 0$, since given any finite n there are certain to be an infinite number of arrivals after n.
So Y_n does not converge almost surely.

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Just applications of what we have learned so far...

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   - Arrival Processes: Discrete time (Bernoulli), Continuous time (Poisson)
   - Markov Chains: Discrete time, Continuous time

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   - Markov Chains: Discrete time, Continuous time
2. Estimation: Bayesian v/s Classical

Stochastic Processes

Suppose our Experiment is to measure $N(t)$ the number of people who leave a bookstore in the interval $[0, T]$. $N(t)$ is a random variable that depends on time. We refer to such an experiment as a "process".

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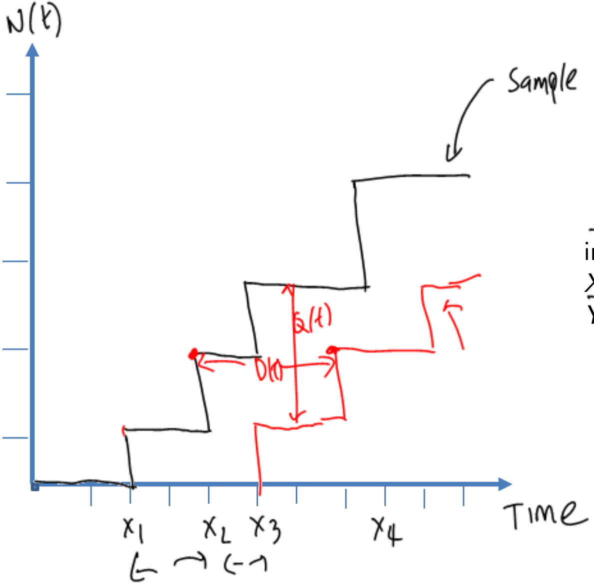
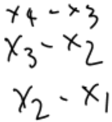
1. The price fluctuations of a stock: How do the future prices depend on past prices?
2. The behavior of a queue over time: What happens to it if we add more cashiers?
3. The sequence of failures times of a machine: Is there a pattern to these failures? Correlation to the failures of other machines in the factory?
4. etc

Arrival Processes

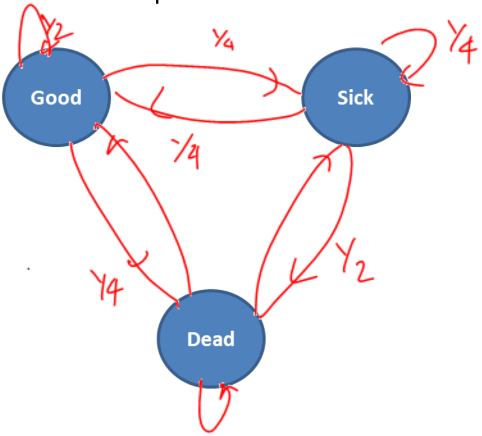


$N(t)$: number of arrivals in $[0, t]$
X_k: time of k^{th} arrival
Y_k: k^{th} interarrival time.



Markov Processes

- Future **depends** on past. However, future value depends only the current state of the system.
- Example: Machine Health



	Good	S	D
Good	1/2	1/4	1/4
Sick			
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3. $T_1 = Y_1$ is the time to the first arrival. $T_2 = Y_2 - Y_1$ is the time between the first and second arrivals. Since coin flipping is memoryless, $Y_2 - Y_1$ is also geometric and independent of Y_1.

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$T_1, T_2, T_3, T_4 \ldots$

Definition 2

A sequence of iid geometric random variables with parameter p.

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View rainy days as "arrivals". Then they must form a Bernoulli Process since the inter-arrival times are iid Geometric.
So both probability that it will rain on any day is p. Answer p^2.

Distribution of the Time of the k^{th} Arrival

Y_k is the sum of the first k interarrival times.
The interarrival times are geometric rvs.
(Geometric Mean: $1/p$; Var: $(1-p)/p^2$.)

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In the first $t-1$ time units we have to had exactly $k-1$ arrivals.
The t^{th} time unit must register an arrival.

In t-1 slots exactly k-1 arrivals

$$P(Y_k = t) = p\binom{t-1}{k-1}p^{k-1}(1-p)^{t-k}$$

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$$\boxed{P(Y_k = t) = \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} p = \binom{t-1}{k-1} p^k (1-p)^{t-k}}$$

Called Pascal Distribution of order k.

Example: Priority System

A packet arrives at a network device at the beginning of each time slot with probability p. It takes 1 slot to process. A Busy/Idle Period is a maximal string of busy/idle slots. Find the distributions of:

BBI BII BI BI BBBI

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3. P[1st and 2nd time he gets an A is on the 8th and 9th Quiz]=

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1. P(He gets exactly two A's in the next six quizzes)= Binomial $k = 2,\ n = 6$.
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3. P[1st and 2nd time he gets an A is on the 8th and 9th Quiz]=

$$P(Y_1 = 8, Y_2 = 9) = P(Y_1 = 8)P(Y_2 - Y_1 = 1) = (1-p)^7 p^2$$

Arriving at a Random Time

Alice has been practicing probability problems. She gets one right with probability p independent of how she has done on the others. Bob walks in and watches Alice do a problem correctly. What is the pmf of the Alice's "success period" which includes this problem?

WRLW(L R)WRRRW

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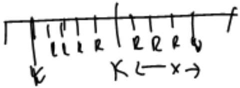

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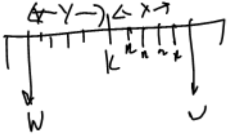

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- We want pmf of $X+Y-1$.
- $X+Y$ is Pascal order 2.

Random Incidence

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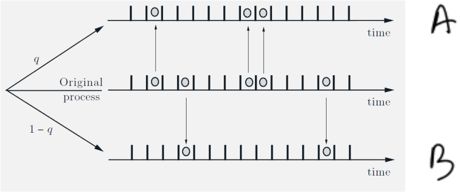
   Expected wait $= 50(0.69) + 5(0.31) \approx 35$ mins.

Splitting Bernoulli Processes

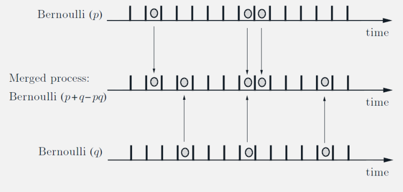


For each arrival, route it to A with prob q and to B with prob $1 - q$. Routing decisions are independent.
A_i: arrival to A at time slot i, and B_i: arrival to B at slot i.

$$P(A_i) = \begin{cases} 1, & \text{if } X_i = 1 \text{ and route to } A; \\ 0, & \text{otherwise.} \end{cases}$$

So, A_i is Bernoulli with prob pq and B_i is Bernoulli with probability $p(1 - q)$.

Merging Bernoulli Processes



A and B are independent Bernoulli Processes, with prob p_A and p_B respectively. Construct the arrival process C as follows:
$C_i = 1$ if $A_i = 1$ or $B_i = 1$ (includes if both have arrivals) and
$C_i = 0$ otherwise.
Then $P(C_i = 0) = (1 - p_A)(1 - p_B)$ and
$P(C_i = 1) = p_C = 1 - (1 - p_A - p_B + p_A p_B) = p_A + p_B - p_A p_B$.
Thus C is Bernoulli Process with probability p_C.

Example: Sum of a geometric number of geometric rvs

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X_i's geometric with p and N is geometric with q
$X_1, X_2, ...$ interarrival times of a Bernoulli process(p).

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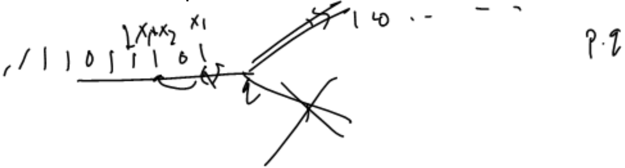

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Construct a split Bernoulli Process as follows:

Each arrival is rejected with prob $1 - q$.
The split Bernoulli has parameter pq. The number of arrivals to the first acceptance is geometric with parameter q. Therefore

$$Y = X_1 + ... + X_N$$

is geometric with parameter pq.

Enjoy Spring Break