EE126: Probability and Random Processes

Lecture 11: Joint PDFs and Conditioning

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Logistics

This week is Travel week...

- Abhay out for the week starting tomorrow
- Guy out all of this week.
- Arash will teach on Thursday and have regular office hours on Friday.

Midterm

- Half a 8.5x11 sheet handwritten both sides
- Calculator
- Tests understanding of the material at the level of the lectures, book and HW.
- Covers the book through 3.3. Also, 4.1 (but only until page 206).

Cumulative Distribution Functions

$$F_x(x) = Pr(X \leq x)$$

Works for continuous and discrete random variables.
The CDF is sometimes referred to as the **Probability Law** of the random variable.

1. Non decreasing, tends to 1 as $x \to \infty$ and to 0 as $x \to -\infty$
2. If X is discrete:

$$p_X(k) = F_X(k) - F_X(k-1)$$

3. If X is continuous

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Exponential Distribution

A non negative random variable X is exponentially distributed with parameter $\lambda > 0$ if

$$f_X(x) = \lambda e^{-\lambda x}$$

- $F_X(x) = 1 - e^{-\lambda x}$
- $E[X] = \lambda \int_0^\infty x e^{-\lambda x} dx = \frac{1}{\lambda}$
- $var(X) = \frac{1}{\lambda^2}$.

Derived Distributions

Suppose we know $F_X(x)$ and want to find $F_Y(x)$ where $Y = g(X)$.

1. Derive the CDF of Y from that of X.
2. Differentiate the CDF to get the pdf.

Example: $F_X(x)$ is uniform on $[a, b]$, $a \geq 0$ and $Y = X^2$.For $y \in [a^2, b^2]$:

$$P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F(-\sqrt{y})$$

$$F_Y(y) = \begin{cases} \frac{\sqrt{y}-a}{b-a}, & 0 \leq a \leq \sqrt{y} \leq b \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \frac{dF_Y(ty)}{dy} = \frac{d(F_X(\sqrt{y}))}{dy} = \frac{1}{2(b-a)\sqrt{y}}$$

Derived Distribution for Linear Functions

If $Y = aX + b$, $a \neq 0$

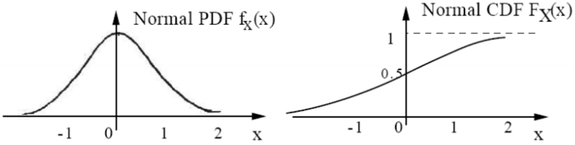
$$f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a})$$

Example: $X \sim$Exponential with parameter λ, $Y = aX + b$

$$f_y(y) = \frac{\lambda}{|a|} e^{-\frac{\lambda}{a}(y-b)}$$

Not an exponential rv, unless $b = 0$ and $a > 0$.

Normal/Gaussian Random Variable



X is a **standard normal** variable (a bell curve with zero mean and unit variance) if

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}, \quad x \in \Re$$

Given any $\mu \in \Re$ and $\sigma^2 > 0$, the Normal/Gaussian R.V. X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$E[X] = \mu$ and $var(X) = \sigma^2$.

Linear function of a Gaussian: Suppose $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ Then $Y \sim N(b + a\mu, a^2\sigma^2)$.

Gaussian RV: Computing Probabilities

Example: Let the temperature of a city in the winter be X in Celsius. $X \sim N(2, 16)$. Find $P(-2 < X < 6)$.
Want to avoid computing the hairy integrals involved...Closed form of Normal CDF not available.
Procedure:

1. Convert to a standard normal: $\hat{X} = \frac{X-2}{4}$
2. Express in terms of probability problem of $\hat{X}$:

$$P(-2 < X < 6) = P(\frac{-2-2}{4} < \hat{X} < \frac{6-2}{4}) = P(-1 < \hat{X} < 1)$$

3. Look it up in a table of computed values of $F_{\hat{X}}(\hat{x})$
   We want to find $P(\hat{X} < 1) - P(\hat{X} < -1) = \phi(1) - \phi(-1)$. Tables typically only list positive values, so $\phi(-1) = 1 - \phi(1)$ by symmetry.
4. Answer: $2\phi(1) - 1 =$

Mixed Random Variables

Suppose X is distributed like r.v. X_1 with probability p_1, like X_2 with prob p_2,...,like X_n with prob p_n. $\sum_i p_i = 1$. Then

$$f_X(x) = \sum_{i=1}^{n} p_i f_{X_i}(x).$$

$$E[X] = \sum_{i=1}^{n} p_i E[X_i].$$

Now suppose some of the n random variables are discrete and some are continuous. The X is a **mixed** random variable.
Important to be very clear about what is discrete and what is not...

Example: Mixed Random Variables

Alice walks to a meeting point where she will either take a taxi or the bus. With prob 2/3 she finds a taxi upon arrival and takes it. Otherwise she waits for the next taxi or bus to arrive, whichever comes first. The next bus will arrive in 5mins. The next taxi will arrive in a time uniform in [0,10]. Find the pdf and cdf of her waiting time, W. Let T be the time for the next taxi. Let A be the event that $T \leq 5$. Then $P(A) = 0.5$ and the r.v. $T|A$ is uniform on $[0, 5]$.
W is a mixed random variable. With prob 2/3 her waiting time is zero. With prob $\frac{1}{6}$ her waiting time is 5, and with prob $\frac{1}{6}$ her waiting time is distributed as $T|A$.

$$f_W(w) = \frac{2}{3}\delta(w) + \frac{1}{6}\delta(w-5) + \frac{1}{6}\frac{1}{5}$$

$$F_W(w) = \begin{cases} \frac{2}{3} + \frac{x}{30}, & 0 \leq x < 5; \\ 1 & x = 5 \end{cases}$$

$$E[W] = \frac{2}{3}(0) + \frac{5}{6} + \frac{5}{12} = \frac{5}{4}$$

Joint PDFs

X, Y are jointly continuous random variables if there is a non-negative function $f_{X,Y}$ such that

$$P(X, Y) \in B = \int_{(x,y)\in B} f_{X,Y}(x, y)dxdy$$

for every subset B of $\Re^2$. and

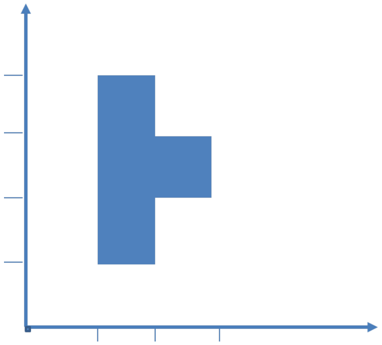
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y)dxdy = 1$$

Properties Carry over

- Marginal of X: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y} dy$,
- Marginal of Y is $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y} dy$
- Independence: X, Y are independent if $f_{X,Y}(x, y) = f_X(x) f_Y(y)$.
- Expected value: $E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dxdy$
- If X and Y are independent then $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$.
- If X and Y are independent then $var(X + Y) = var(X) + var(Y)$.

Example

$f_{XY}(x, y) = c$ over shaded region, zero elsewhere. Find c and both marginals.



Joint CDF

$$F_{X,Y}(X,Y) = P(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)dxdy$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

Example: $Z = \max\{X, Y\}$:

$$F_Z(z) = P(\max\{X,Y\}) \leq z = P(X \leq z, Y \leq z) = F_{X,Y}(z,z)$$

X, Y iid Uniform on $[0,1]$. Then $F_{X,Y}(z,z) = F_X(z)F_Y(z) = z^2$, Thus

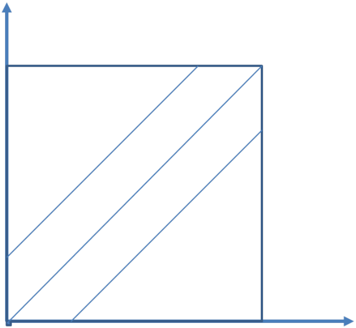
$$f_Z(z) = \begin{cases} 2z, & z \in [0,1]; \\ 0, & \text{o.w.} \end{cases}$$

Example: IID Uniform Random Variables

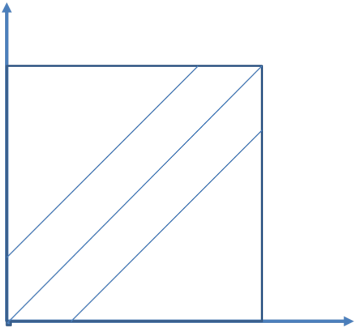
When X, Y are iid uniform on $[0, 1]$, all pairs of outcomes are equally likely. Often easier to calculate **areas** directly instead of integrals.

1. $Z = |X - Y|$: Recall the Alice-Bob problem:
2. $Z = X - Y$:
3. $Z = X + Y$
4. Break a stick "uniformly" at two points. What is the prob the pieces form a triangle?

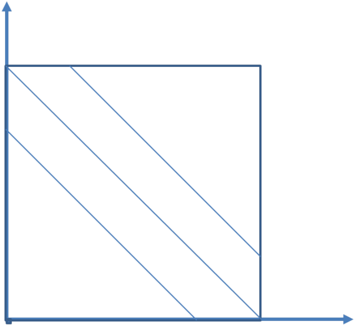
$Z = |X - Y|$



$Z = X - Y$:



$Z = X + Y$

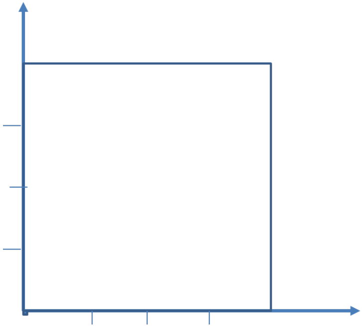


$$P(Z \le z) = \begin{cases} \frac{1}{2}z^2, & 0 \le z \le 1; \\ 1 - \frac{1}{2}(2-z)^2, & 1 < z \le 2. \end{cases}$$

$$f_Z(z) = \begin{cases} z & 0 \le z \le 1; \\ 2 - z, & 1 < z \le 2. \end{cases}$$

Stick Breaking -I

Break a stick "uniformly" at two points, x, y:
The three pieces are of length $\min\{x, y\}, \max\{x, y\}, 1 - \max\{x, y\}$.



$Z = |X - Y|$ revisited

Standard Method

$$Pr(Z \le x) = P(|X - Y| \le z) \le P(-z \le X - Y \le z)$$

Let $W = X - Y$ and find $F_W(w)$, $-1 \le w \le 1$.

$$F_Z(z) = F_W(z) - F_W(-z) \quad z \in [0, 1]$$

Now

$$P(W \le w) = \begin{cases} 0.5 + (.5 - .5(1-w)^2), & 0 \le w \le 1; \\ 0.5(1+w)^2 & -1 \le w < 0. \end{cases}$$

$$P(Z \le z) = 0.5 + (.5 - .5(1-z)^2) - 0.5(1-z)^2$$

$$f_Z(z) = (1 - z) + (1 - z) = 2(1 - z)$$

Conditioning on an Event

Carries over from the discrete case:
Given any subset B of the real line and an event A:

$$P(X \in B|A) = \int_B f_{X|Y}(x)dx$$

If the event, A is a subset of the real line:

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)}, & \text{if } x \in A; \\ 0, & \text{o.w.} \end{cases}$$

Example: Residual life of X: Given that $X > t$, what is the dist of residual life, R?

$$P(R > r|X > t) = P(X > t + r|X > t) = \frac{P(X > t + r)}{P(X > r)}$$

For X uniform $[0, 1]$, $\frac{1-t-r}{1-t} = 1 - \frac{r}{1-t}$.

Conditioning on a Random Variable

Given continuous X, Y, suppose we want $P(X \leq x | Y = y)$. Now clearly, $P(Y = y) = 0$, so what does this mean?

Example: Radar detector error $X \sim N(0, y/10)$ where Y is the speed of the car. Y is also a random variable.

Need a definition for $f_{X|Y}$: Consider the event $B = \{y \leq Y \leq y + \delta_y\}$.

$$P(x \leq X \leq X + \delta_x | B) \approx \frac{f_{X,Y}(x,y)\delta_x\delta_y}{f_Y(y)d_y} = \frac{f_{X,Y}(x,y)\delta_x}{f_Y(y)}$$

Define

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

$$P(x \leq X \leq X + \delta_x | B) \approx f_{X|Y}(x|y)\delta_x$$

Since δ_y is independent of δ_x, we can let δ_y go to zero.

$$P(x \leq X \leq X + \delta_x | Y = y) \approx f_{X|Y}(x|y)\delta_x$$