EE126: Probability and Random Processes

Lecture 1: Probability Models

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Practical Information

- Teaching Staff: Instructor: Prof. Abhay Parekh,
  GSIs: Guy Bresler and Arash Ali Amini
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- Lectures: Tuesday, 3:30–5:00 PM, Thursday, 3:30–5:00 PM,3106 Etcheverry.
- Discussions:Tues, 5:00–6:00 PM, 299 Cory. Thur, 500-6:00 PM, 241 Cory
- Office Hours: Parekh: 514 Cory TTh10-11,
  Bresler: TBD Office Hours: TBA,
  Amini: TBD, Office Hours: TBD

Logistics

- Textbook: Introduction to Probability by Bertsekas and Tsitsiklis. Available at the campus book store. In addition to attending lectures and discussions, doing problems and reading the textbook outside class will be an integral part of the learning process.
- Pre-requisites: EE 20, and MATH 53/54 (multivariate calculus; linear algebra) or equivalent. (Minor overlap with part of CS70)
- Course Website: On bspace. Maybe Piazzza for questions. Lecture notes on or slightly before day of lecture. Readings for week posted. **This week: 1.1-1.4.**

Grading

- Grade: Homework (15%), two midterms (20% each), and one final exam (45%). *All exams are cumulative in nature.* Midterm 1: TBD Midterm 2: TBD Final Exam: TBD
- Homeworks: Problem sets will be posted on bspace (roughly one per week), and will be due in class on the date specified on the problem set. *Late homeworks will not be accepted.*
- After attempting the problems on your own, you can discuss homework assignments in groups of at most three. However, you must *write up your own solutions* individually, and must explicitly name any collaborators at the top of the homework.

What is this course about?

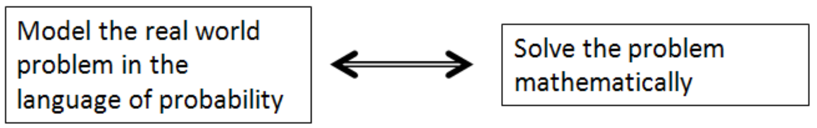
Most real-world problems involve uncertainty

1. Predictions:
   - Will you like this class? Will you get an A ?
   - Will the Giants win the world series next year? What are the odds?
2. Strategy/Decision Making
   - How should you bet in blackjack?
   - Should you buy shares of AAPL?
3. Engineering
   - How should you build a spam filter?
   - How should you build a wifi system?

Probability is a mathematical discipline that allows one **to reason about uncertainty**

What is this course about?

Two equally important skills taught in this class



Understand the problem as an experiment

Combinatorics, Calculus

Many students get stuck on the first block. Solve lots of problems...

Basic Framework: The Experiment

Concept

1. The experiment has **base outcomes**
2. These outcomes are exhaustive and mutually exclusive
3. Each base outcome, e is assigned a non-negative number $P(e)$ called its probability
4. The probabilities summed over all of the base outcomes always equals 1

Example: Toss a fair coin twice

1. Base outcomes are HH,HT,TH,TT
2. This covers **all** the possibilities. **M.E.**
3. Each of these outcomes is equally likely
4. Assign each outcome a probability of 0.25.

The list (set) of base outcomes is called the **Sample Space**.
The rules by which probabilities are assigned are the **Axioms of Probability.**

Sample Space, Ω

1. Toss a Coin 3 times

2. Pick 3 balls from an urn with 2 Red and 3 Black Balls

3. Toss a coin until the first H appears

Event: A(ny) subset of the Sample Space

1. E_1: Get 2 Heads when experiment is to toss a Coin 3 times

2. E_2: Get 1 Black and 2 Reds when exp is to pick 3 balls from an urn with 2 Red and 3 Black Balls

3. E_3: Even number of tosses when exp is to toss a coin until the first H appears

4. **Always events:** The base outcomes; The set of all base outcomes, Ω and the null outcome $\{\}$.

The Axioms of Probability

Probabilities are assigned to EVENTS

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. If $A_1, A_2, \ldots,$ are mutually exclusive

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Set Theory Can be Useful

Events don't have to be mutually exclusive.

1. If $A \subset B$ then $P(A) \leq P(B)$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
3. $P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$ [in recitation]
4. $(A \cup B)^c = A^c \cap B^c$
5. etc

Discrete Uniform Law

Suppose all the base outcomes of an experiment are equally likely

Given any event A:

$$P(A) = \frac{\text{Number of base outcomes in } A}{\text{Total number of base outcomes}} = \frac{|A|}{|\Omega|}.$$

Two Die Toss

Experiment: Toss 2 six sided dice.

(T_1, T_2)	1	2	3	4	5	6
1	.	.	.	.	.	.
2	.	.	.	.	.	.
3	.	.	.	.	.	.
4	.	.	.	.	.	.
5	.	.	.	.	.	.
6	.	.	.	.	.	.

Each base outcome has probability 1/36.

- $P(T_1 = 1) =$
- $P(T_1 = T_2) =$
- $P(\min\{T_1, T_2\} = 3) =$

Toss to First Head Example Revisited

$\Omega = \{H, TH, TTH, TTTH, TTTTH, ...\}$

- Let $E_i =$ Heads on the i^{th} toss.
- What should be $P(E_i)$? Has to add up to 1
- What about $P(E_i) = \frac{1}{i}$? Adds to ∞.
- Correct answer (we will see why later) is $P(E_i) = \frac{1}{2^i}$.

$$\begin{aligned} P(\text{even number of tosses}) &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \end{aligned}$$

Why not $\frac{1}{2}$?

Example: Monty Hall Problem

Game Show: 3 doors. Awesome prize behind one of them. Contestant picks a door. Monty opens **one of the other** two doors that does not have a prize. Contestant can either stick to the original door or change selection. What should he/she do?? Can switching help?

W : contestant wins; A : picks prize door; S : decides to switch.

$$P(W) = P(A \cap S^c) + P(A^c \cap S)$$

If $P(S) = 0 : P(W) = P(A) = \frac{1}{3}$
If $P(S) = 1 : P(W) = P(A^c) = \frac{2}{3}$.
Surprisingly, it is much better to switch!

Controversy

The problem appeared in "Parade" Magazine in a column by Marilyn Vos Savant.

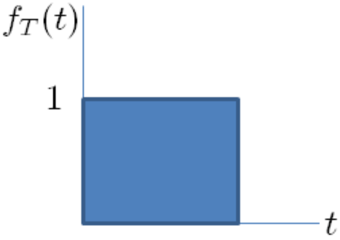
From the Wikipedia

Though vos Savant gave the correct answer that switching would win two-thirds of the time, she estimates the magazine received 10,000 letters including close to 1,000 signed by PhDs, many on letterheads of mathematics and science departments, declaring that her solution was wrong (Tierney 1991). As a result of the publicity the problem earned the alternative name Marilyn and the Goats.

Continuous Sample Spaces

Experiment: At bus stop; time to next bus.

- Suppose the bus will definitely arrive in ≤ 1 hour.
- Then $\Omega = \{t \in (0,1]\}$.
- Assigning probabilities becomes a bit tricky...
- Example: If bus is equally likely to come at any time in $(0,1]$, then $P(t = .5) = 0$
- Solution: Define a function $f_T(t)$ such that $\int_0^1 f_T(t)dt = 1$



$P(\alpha \leq t \leq \beta) = \int_\alpha^\beta dt = \beta - \alpha$ for $0 < \alpha, \beta \leq 1$.

Continuous Sample Spaces

- In all engineering problems you have finite precision so the sample space is discrete
- Continuous sample spaces are merely approximations that make calculations and understanding easier
- Don't worry about continuous sample spaces right now, but now may be a good time to brush up on your calculus!